Running Example: Dumbbell Code

$$H := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} C = \{ \; & (0,0,0,0,0,0,0), (1,1,1,0,0,0,0), \\ & (0,0,0,0,1,1,1), (1,1,1,0,1,1,1) \; \} \end{aligned}$$

Parity Polytope of Dumbbell Code

$$H := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$\Omega(H)$ is all points of the form $(a, a, a, b, c, c, c) \in \mathbb{R}^7$ with

$$0 \le a \le 1 \quad 0 \le b \le 1 \quad 0 \le c \le 1$$

and

$$b \le 2a \qquad\qquad b \le 2c$$

$$2a + b \le 2 \qquad\qquad b + 2c \le 2$$

i.e.,

$$\frac{b}{2} \le a \le 1 - \frac{b}{2}$$
$$\frac{b}{2} \le c \le 1 - \frac{b}{2}$$

Vertices of $\Omega(H)$

$$\Omega(H) = \left\{ (a,a,a,b,c,c,c) \in \mathbb{R}^7 \;\middle|\; \begin{array}{c} 0 \le a \le 1 \\ 0 \le b \le 1 \\ 0 \le c \le 1 \\ \frac{b}{2} \le a \le 1 - \frac{b}{2} \\ \frac{b}{2} \le c \le 1 - \frac{b}{2} \end{array} \right\}$$

There are five vertices:

$$(0,0,0,0,0,0,0)$$

$$(1,1,1,0,0,0,0)$$

$$(0,0,0,0,1,1,1)$$

$$(1,1,1,0,1,1,1)$$

$$(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})$$

Cost Function Example

Suppose $\tilde{\mathbf{y}} = (0,1,0,1,0,1,0)$ is received (and $\mathbf{0}$ was sent) from a BSC with probability of error 0.1. Then

$$\begin{aligned}\gamma &= (\ln\left(\frac{.9}{.1}\right), \ln\left(\frac{.1}{.9}\right), \ln\left(\frac{.9}{.1}\right), \ln\left(\frac{.1}{.9}\right), \ln\left(\frac{.9}{.1}\right), \ln\left(\frac{.1}{.9}\right), \ln\left(\frac{.9}{.1}\right)) \\ &= (2.198, -2.198, 2.198, -2.198, 2.198, -2.198, 2.198)\end{aligned}$$

We have $d(\tilde{\mathbf{y}}, \mathbf{0}) = 3$ and $d(\tilde{\mathbf{y}}, \mathbf{x}) > 3$ for all $\mathbf{x} \in C \setminus \{\mathbf{0}\}$, but the cost of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is negative and we have a decoding error.

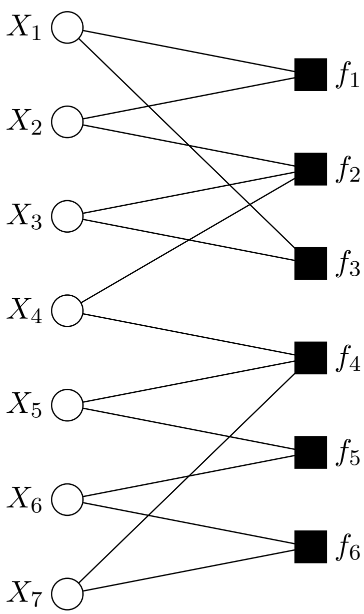
Fundamental Cone

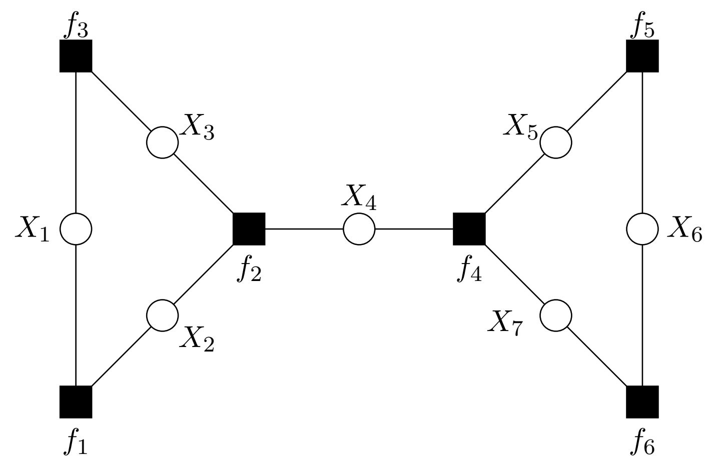
$$H := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{K}(H) = \left\{ (a,a,a,b,c,c,c) \in \mathbb{R}^7 \;\middle|\; \begin{array}{c} a \geq 0, b \geq 0, c \geq 0 \\ 2a \geq b, 2c \geq b \end{array} \right\}$$

Tanner graph

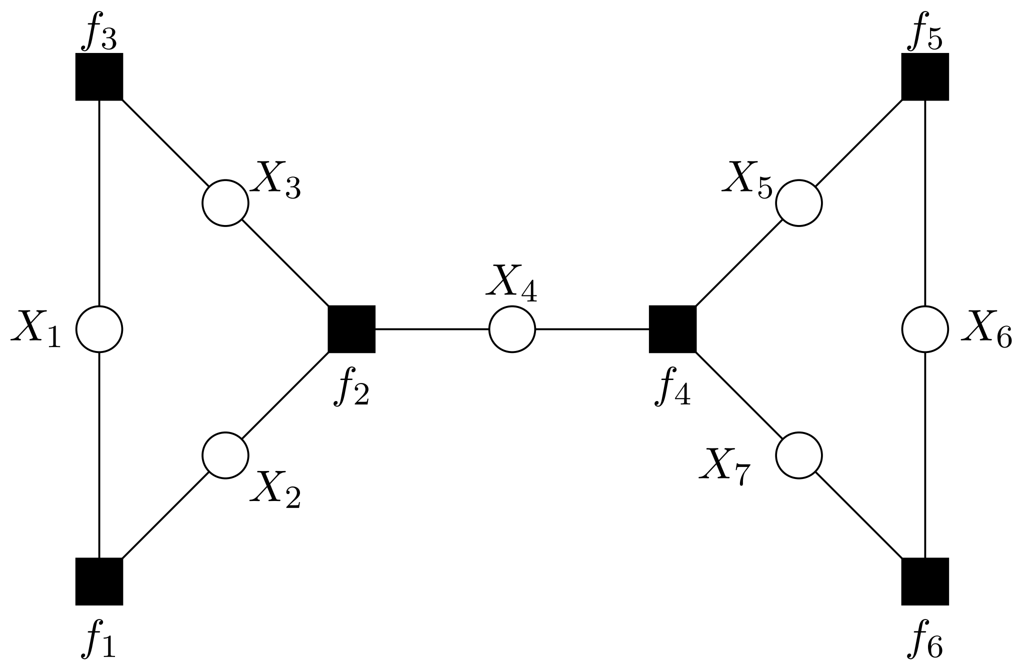
$$H := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



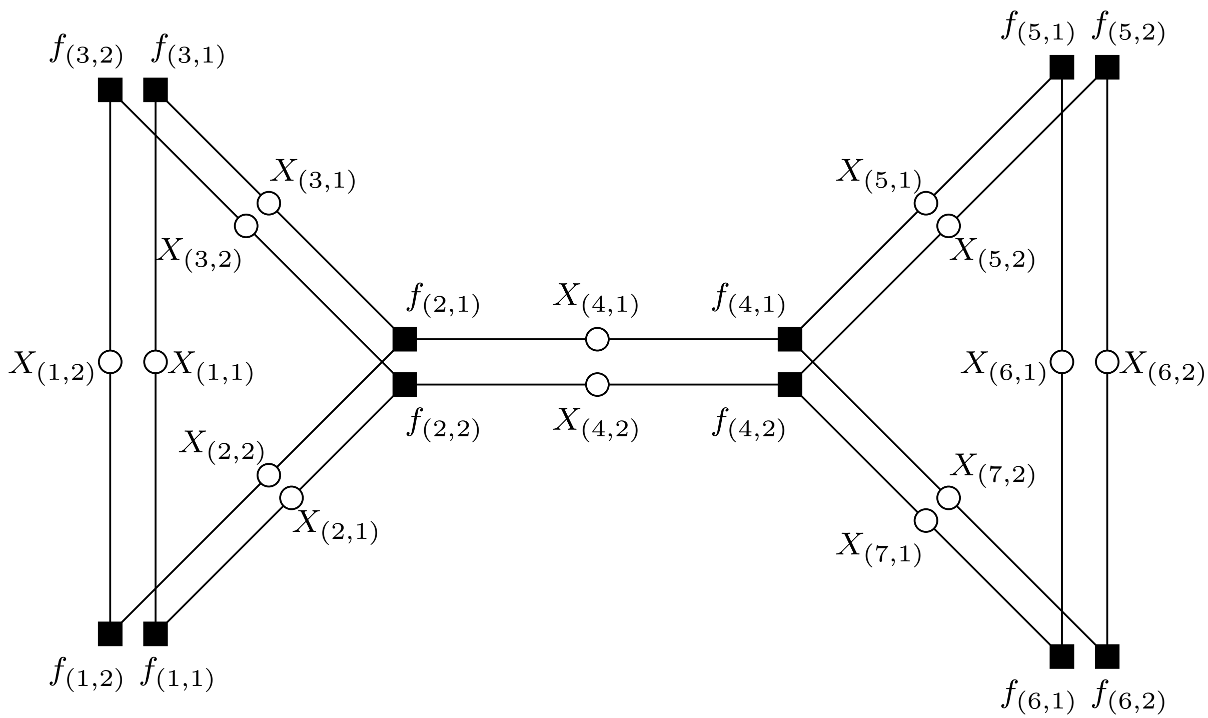


Graphical representation

$$\mathbf{c} = (1, 1, 1, 0, 0, 0, 0)$$

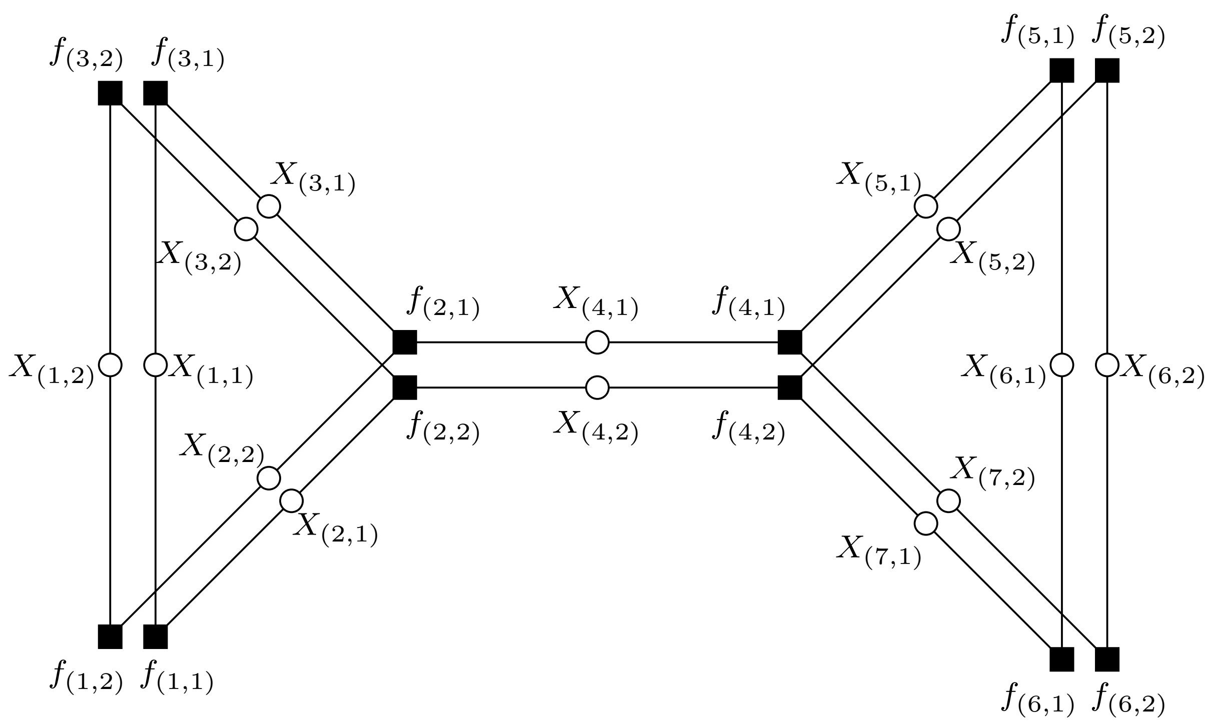


A 2-cover of the Dumbbell Graph



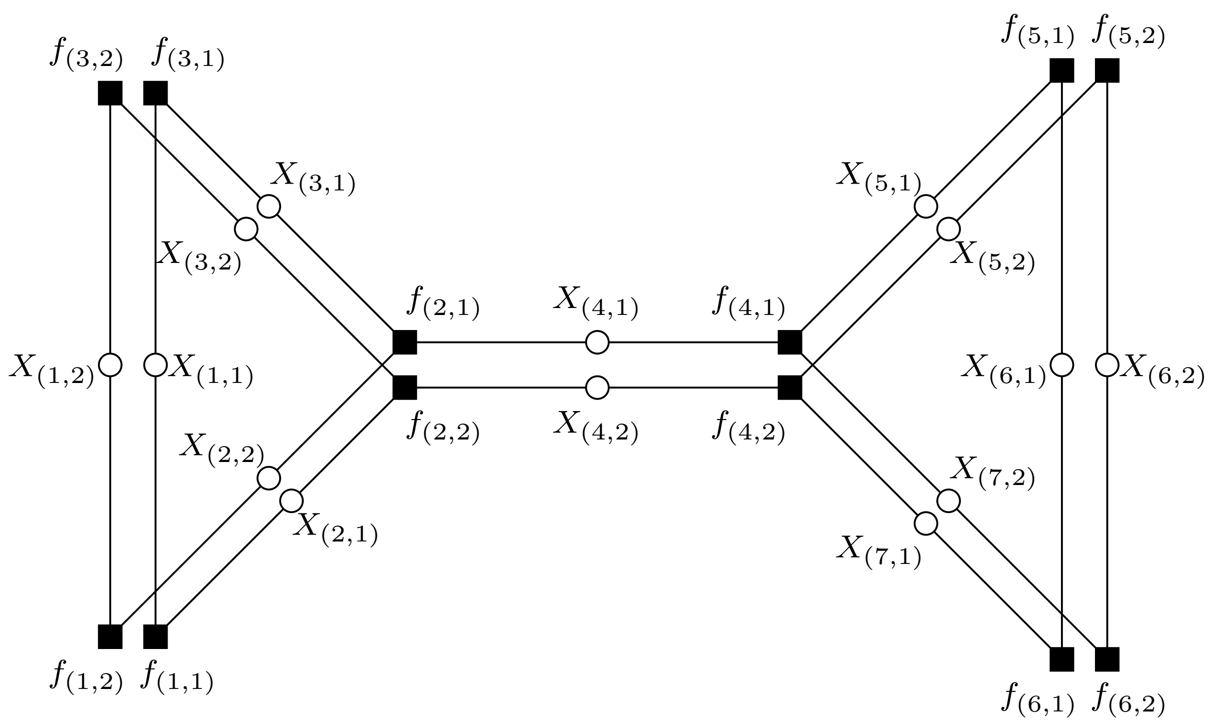
A lift to the 2-cover

$$\tilde{\mathbf{c}} = (1{:}1, 1{:}1, 1{:}1, 0{:}0, 0{:}0, 0{:}0, 0{:}0)$$



A codeword on the 2-cover which is not a lift

$$\tilde{\mathbf{a}} = (1{:}0, 1{:}0, 1{:}0, 1{:}1, 1{:}0, 1{:}0, 1{:}0)$$



Pseudo-codewords

The pseudo-codeword associated to

$$\tilde{\mathbf{c}} = (1{:}1, 1{:}1, 1{:}1, 0{:}0, 0{:}0, 0{:}0, 0{:}0)$$

is

$$\mathbf{p}(\tilde{\mathbf{c}}) = (2, 2, 2, 0, 0, 0, 0).$$

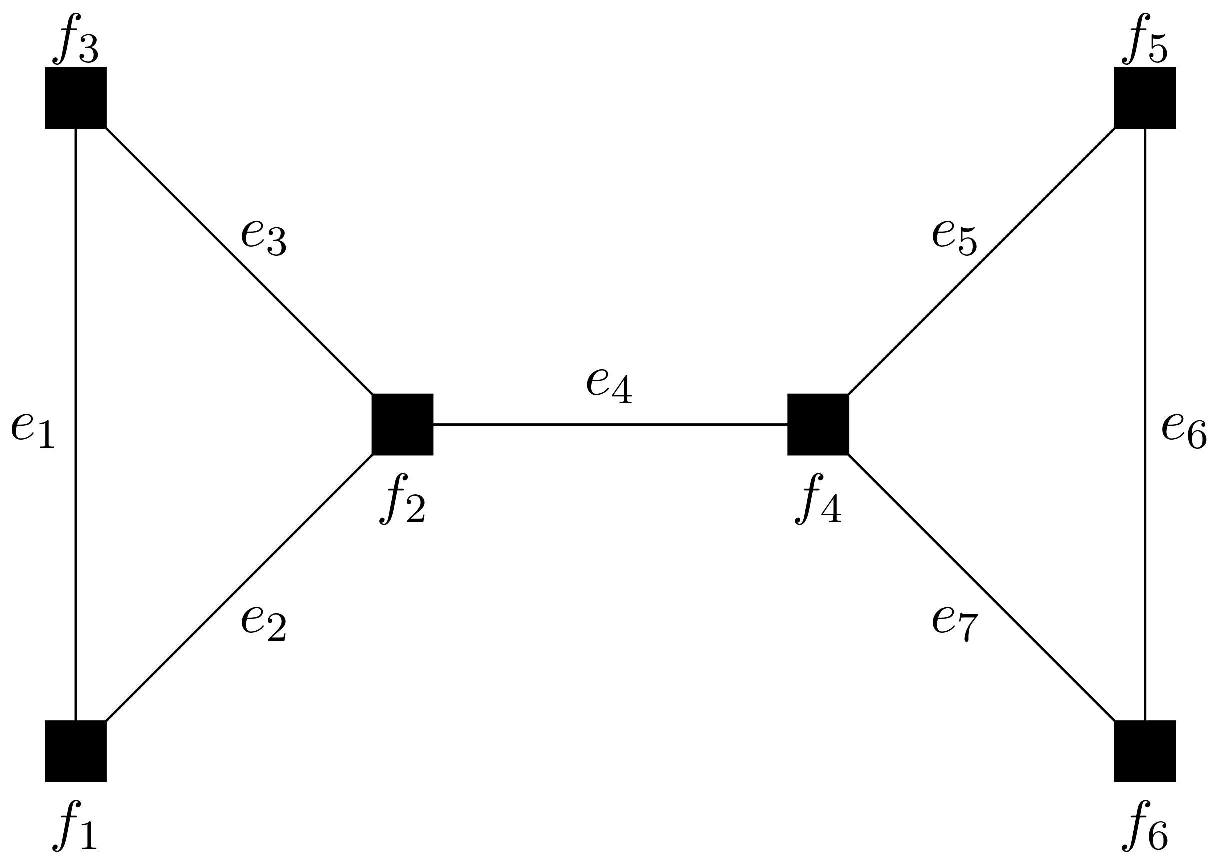
The pseudo-codeword associated to

$$\tilde{\mathbf{a}} = (1{:}0, 1{:}0, 1{:}0, 1{:}1, 1{:}0, 1{:}0, 1{:}0)$$

is

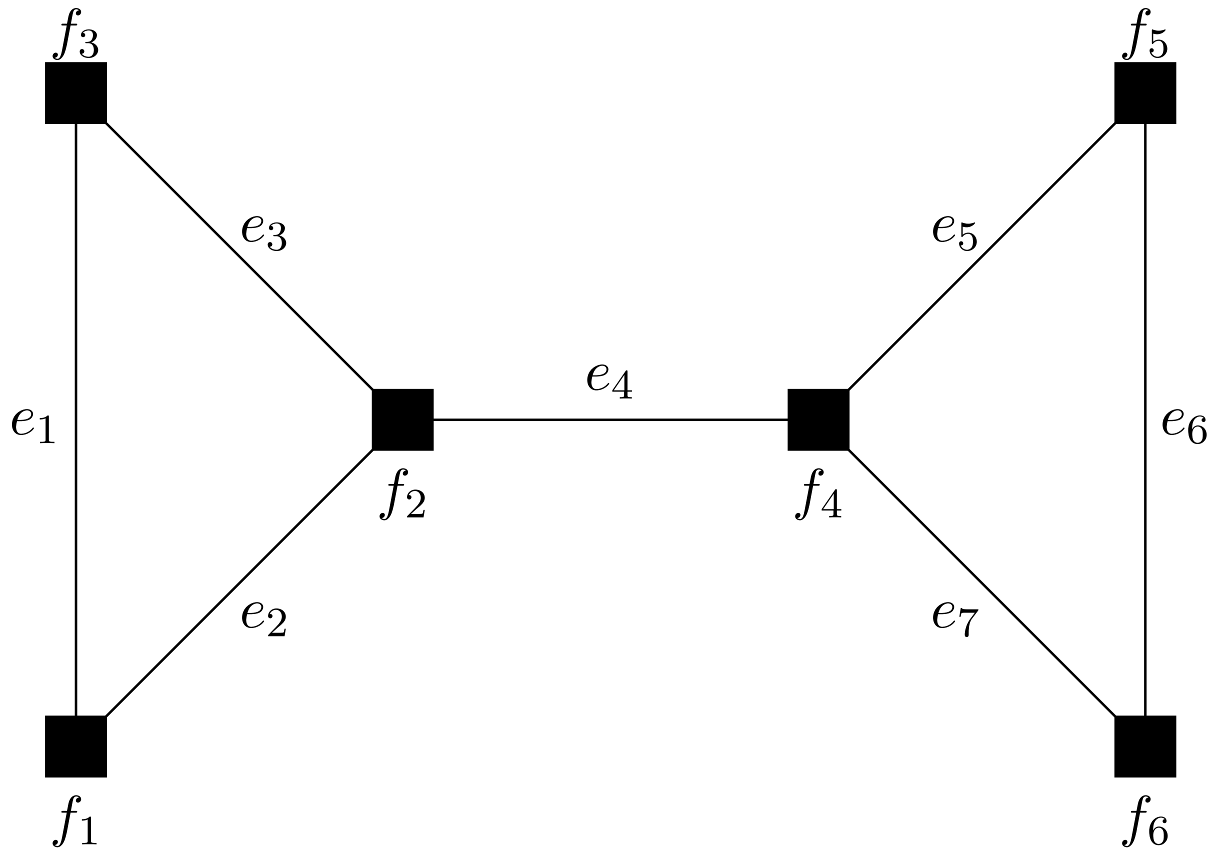
$$\mathbf{p}(\tilde{\mathbf{a}}) = (1, 1, 1, 2, 1, 1, 1).$$

Normal graph

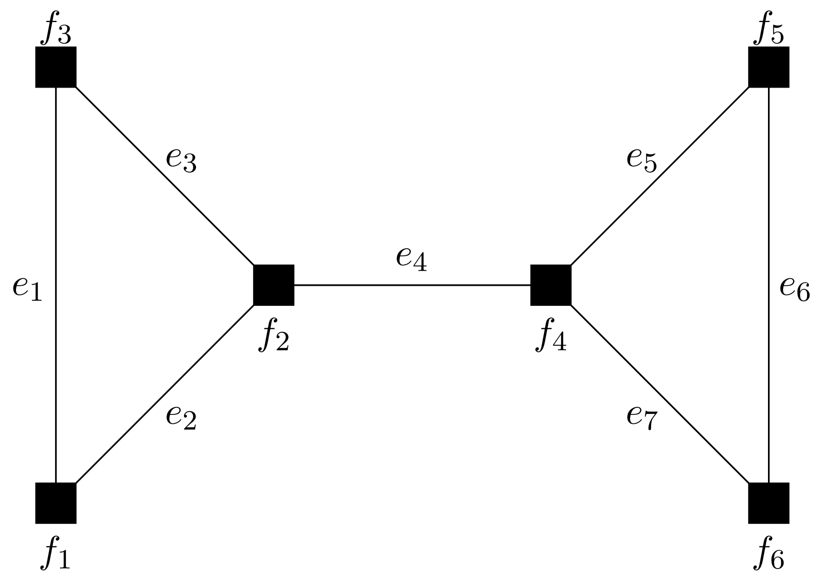


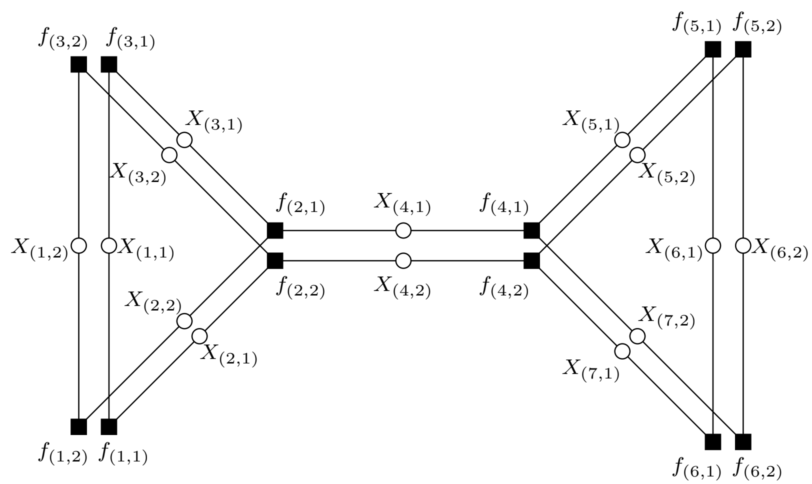
Normal graph

The codeword $(1, 1, 1, 0, 0, 0, 0)$ corresponds to the cycle (e_1, e_2, e_3).



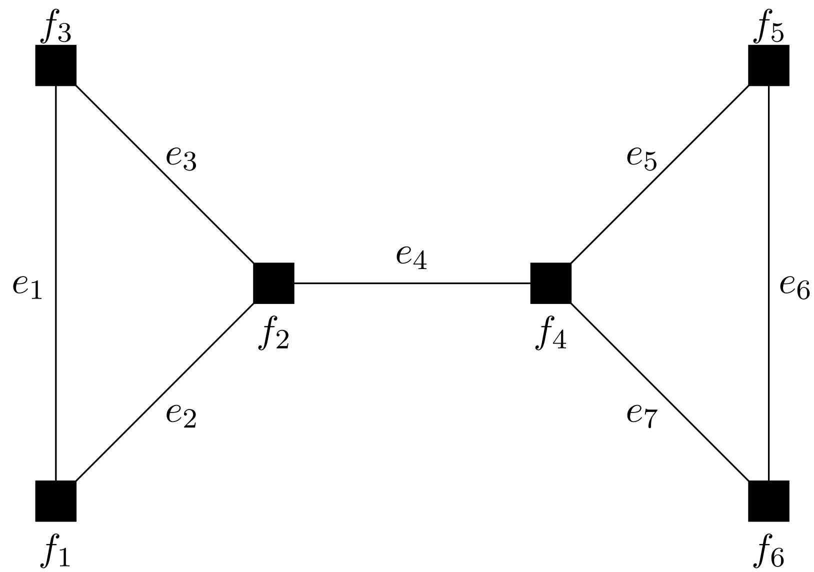
Pseudo-codeword of a cycle code





Monomial of a path

The monomial of the path $(e_1, e_2, e_4, e_5, e_6, e_7, e_4, e_3)$



is $u_1 u_2 u_3 u_4^2 u_5 u_6 u_7$.

Edge zeta function for Dumbbell Code

Let N be the normal graph for the Dumbbell Code. Then

$$\zeta_N(u_1,\ldots,u_7)^{-1} = 1 - 2u_1u_2u_3 + u_1^2u_2^2u_3^2 - 2u_5u_6u_7 \\ + 4u_1u_2u_3u_5u_6u_7 - 2u_1^2u_2^2u_3^2u_5u_6u_7 - 4u_1u_2u_3u_4^2u_5u_6u_7 \\ + 4u_1^2u_2^2u_3^2u_4^2u_5u_6u_7 + u_5^2u_6^2u_7^2 - 2u_1u_2u_3u_5^2u_6^2u_7^2 \\ + u_1^2u_2^2u_3^2u_5^2u_6^2u_7^2 + 4u_1u_2u_3u_4^2u_5^2u_6^2u_7^2 - 4u_1^2u_2^2u_3^2u_4^2u_5^2u_6^2u_7^2.$$

Edge zeta function for Dumbbell Code II Expanding out the Taylor series gives

$$\begin{aligned}\zeta_N(u_1,\ldots,u_7) &= 1 + 2u_1u_2u_3 + 3u_1^2u_2^2u_3^2 + 2u_5u_6u_7 + 4u_1u_2u_3u_5u_6u_7\\ &\quad + 6u_1^2u_2^2u_3^2u_5u_6u_7 + 4u_1u_2u_3u_4^2u_5u_6u_7 + 12u_1^2u_2^2u_3^2u_4^2u_5u_6u_7\\ &\quad + 3u_5^2u_6^2u_7^2 + 6u_1u_2u_3u_5^2u_6^2u_7^2 + 9u_1^2u_2^2u_3^2u_5^2u_6^2u_7^2\\ &\quad + 12u_1u_2u_3u_4^2u_5^2u_6^2u_7^2 + 36u_1^2u_2^2u_3^2u_4^2u_5^2u_6^2u_7^2 + \cdots .\end{aligned}$$

More precisely . . .

- Label the bit nodes of T as $x_1, \ldots, x_n$
- Label the edges of T as $e_{(i,j)}$, $1 \le i \le n$ and $1 \le j \le d_i$, so that $e_{(i,1)}, \ldots, e_{(i,d_i)}$ are incident to x_i
- Set $N := \sum d_i = \#$ of edges of T
- Define $\phi : \mathbb{F}_2^N \to \mathbb{F}_2^n$ by

$$\phi((\mathbf{c})_{(i,j)}) = (c_{(1,1)}, c_{(2,1)}, \ldots, c_{(n,1)})$$

- $\widehat{C}' = \{\mathbf{c} \in \widehat{C} \mid c_{(i,1)} = \cdots = c_{(i,d_i)} \text{ for all } i\}$

Then $C = \phi(\widehat{C}')$.