Space-Time Block Codes from Cyclic Division Algebras

Codes, Encoders and Decoders

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Sequences and Codes
SFU, Vancouver, BC

Part I

Introduction

Outline

Paths recombination

- Each path is characterized by its magnitude α_i, its phase θ_i and its delay, τ_i.

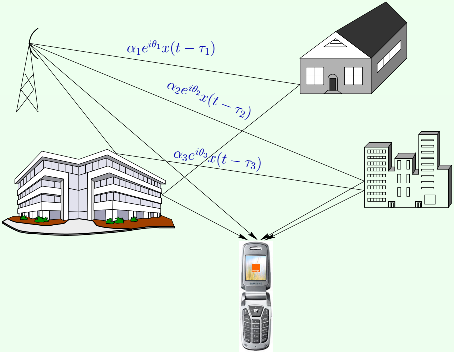


Figure: Destructive recombination due to phases $\longrightarrow$ *fadings* (here, $x(t)$ is the transmitted signal)

Phases dependencies

Phase terms

$$\theta_i = -2\pi f_0 \tau_i$$

where f_0 is the carrier (or subcarrier) frequency.

- Fadings vary as a function of
  - frequency
  - antennas position (since τ_i are different from one antenna to the other one)
  - time (obstacles and terminals may move)
  - ...

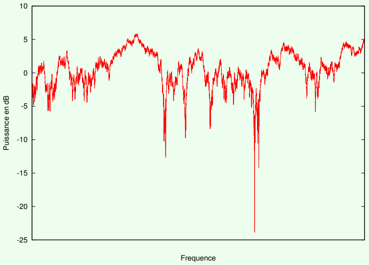


Figure: Received power as a function of the frequency

Outline

Diversity

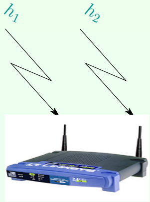
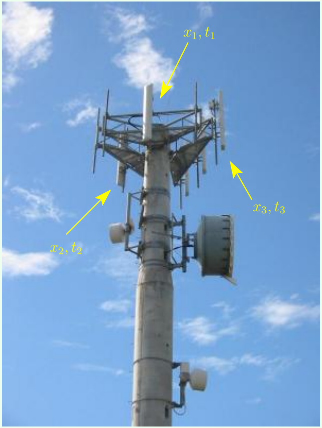


Figure: Access Point 802.11g : Selection of antenna 1 *iff* $|h_1|^2 > |h_2|^2$

- **Space diversity:** Selection or Maximum Ratio Combining
- **Time or Frequency or Space-Time diversity:** Same techniques, but transmitter must introduce redundancy on different time slots, frequencies, or antennas $\rightsquigarrow$ bad resource use.

Space-Time diversity



- Example of a GSM Base station
- At each time burst t_i, antenna i transmits burst x_i.
- Antennas are assumed spaced enough so that fadings can be assumed independent

Space-Time Coding

It is a primitive example of space-time coding

Outline

Error Correcting Codes vs Space-Time Codes

Error correcting codes

Codes defined on a finite alphabet (usually $\mathbb{F}_{2^m}$) $\in$ **Binary World**.

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Codes defined on $\mathbb{C}$. These codes act as an interface $\in$ **Signal space World**.

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Codes defined on $\mathbb{C}$. These codes act as an interface $\in$ **Signal space World**.

1. Inner "code" $\longrightarrow$ Space-Time Block Code
2. Outer "code" $\longrightarrow$ Binary (or non binary) code (Convolutional, Block, LDPC, Turbo,...)

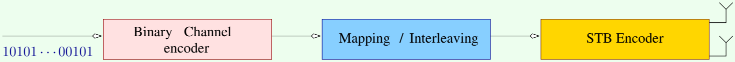

Figure: The MIMO communication chain

Part II

The MIMO Channel

Outline

The quasi static fading channel

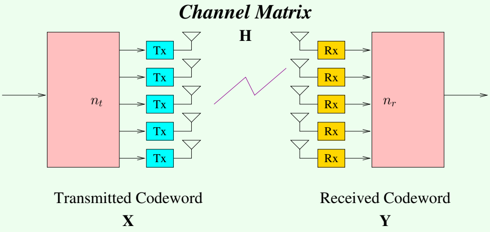

Figure: The Channel Model

- Received signal

$$\mathbf{Y}_{n_r \times T} = \mathbf{H}_{n_r \times n_t} \cdot \mathbf{X}_{n_t \times T} + \mathbf{W}_{n_r \times T} \quad (1)$$

  with **H** perfectly known at the receiver (coherent case).
- **H** is assumed constant during the transmission of one codeword.

Pairwise Error Probability

- Pairwise error probability for a quasi-static Rayleigh fading channel is upper bounded by

$$P(\mathbf{X_1} \to \mathbf{X_2}) \leq \left(\prod_{i=1}^{n_t} \frac{1}{1+\lambda_i^2 \frac{E_s}{4N_0}} \right)^{n_r} \quad (2)$$

with λ_i^2 being the eigenvalues of $(\mathbf{X}_1 - \mathbf{X}_2)^\dagger \cdot (\mathbf{X}_1 - \mathbf{X}_2)$ counting the multiplicities

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- **Two criteria**
  - **The rank criterion** : In order to achieve maximum diversity $n_t \cdot n_r$, the matrix $(\mathbf{X}_1 - \mathbf{X}_2)$ must be of maximum rank n_t.
  - **The coding advantage** : In order to maximize the coding gain, the quantity

$$\min_{\mathbf{X_1} \neq \mathbf{X_2}} \det\left(\mathbf{X}_1 - \mathbf{X}_2\right)^\dagger \cdot \left(\mathbf{X}_1 - \mathbf{X}_2\right) \tag{3}$$

must be maximized.

Outline

The Gaussian MIMO channel

- Denote $\mathbf{Q}$ the covariance matrix of the Gaussian channel input $\mathbf{X}$,

$$\mathbf{Q} = \mathbb{E}\left[\mathbf{X}^{\dagger}\mathbf{X}\right].$$

Then, mutual information is

$$I(\mathbf{X};\mathbf{Y}) = \log_2 \det\left(\mathbf{I}_{n_r} + \frac{1}{\sigma^2}\mathbf{H}^{\dagger}\mathbf{Q}\mathbf{H}\right)$$

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If we choose a channel entry such that $\mathbf{Q} = \frac{P_{\mathbf{X}}}{n_t}\mathbf{I}_{n_t}$, then we get,

$$I(\mathbf{X};\mathbf{Y}) = \log_2 \det\left(\mathbf{I}_{n_r} + \frac{\gamma}{n_t}\mathbf{H}^{\dagger}\mathbf{H}\right) \tag{4}$$

The ergodic capacity

- The channel varies fast enough so that the temporal and probabilistic averages are equal.
  - It does not correspond to a practical case
    - The channel must vary very fast
    - No delay constraint

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In this case, we get the ergodic capacity

$$\begin{aligned} C(\gamma) &= \mathbb{E}_{\mathbf{H}}\left[\log_2 \det\left(\mathbf{I}_{n_r} + \frac{\gamma}{n_t}\mathbf{H}^{\dagger}\mathbf{H}\right)\right] \\ &= \min\{n_t, n_r\}\log_2 \gamma + O(1) \end{aligned} \quad (5)$$

where $\min\{n_t, n_r\}$ is the number of degrees of freedom of the channel.

The outage probability (I)

- Consider $C(\mathbf{H})$ as the optimized mutual information for the MIMO Gaussian channel,

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- We consider $C(\mathbf{H})$ as a random variable (Block Fading Channel)
- The outage probability is

$$P_{\text{out}} = \Pr\{C(\mathbf{H}) \leq R\}$$

where R is the binary rate of the system.

The outage probability (II)

Asymptotic behavior

For a high SNR γ, we get

$$P_{\text{out}} = O\left(\frac{1}{\gamma^d}\right)$$

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where d is the physical diversity of the system.

- The physical diversity is the maximal obtainable order of diversity
- For a quasistatic channel, and decorrelated transmit and receive antennas, we have

$$d = n_t \cdot n_r$$

The Diversity vs. Multiplexing Gain tradeoff (I)

- We have seen two main results
  - For ergodic channels, the capacity is asymptotically equal to

$$C(\gamma) = \min(n_t, n_r)\log_2 \gamma + O(1)$$

where $r = \min(n_t, n_r)$ is the number of degrees of freedom or the multiplexing gain of the system

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- We are interested in block fading channels. Can we have both the diversity and the multiplexing gain ?

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YES...but [Zheng Tse 2003]

We can both gain on r and d but there is a fundamental tradeoff between the two gains.

The Diversity vs. Multiplexing Gain tradeoff (II)

Definitions

Define the spatial multiplexing gain and the diversity gain

$$r = \lim_{\gamma \to \infty} \frac{R(\gamma)}{\log_2 \gamma} \quad d = -\lim_{\gamma \to \infty} \frac{\log P_e(\gamma)}{\log(\gamma)}.$$

Equivalently, we can write (exponential equality)

$$P_e(\gamma) \doteq \gamma^{-d}$$

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Theorem

The optimal tradeoff curve $d^*(r)$ is a piecewise-linear function connecting the points $d^*(k)$ for $k = 0, 1, \cdots, \min\{n_t, n_r\}$ where

$$d^*(k) = (n_t - k)(n_r - k)$$

The Diversity vs. Multiplexing Gain tradeoff (III)

$n_t = n_r = 4$

D-MG Tradeoff. Remark the extremal values: Maximum diversity 16 and maximum multiplexing gain 4

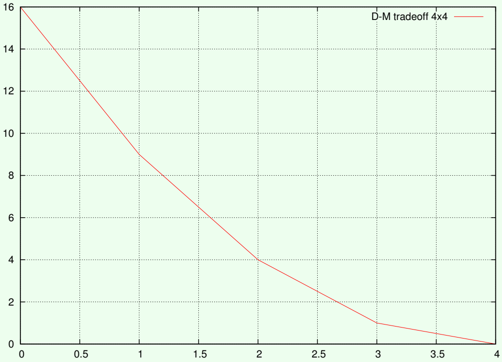

Figure: The D-MG tradeoff for $n_t = n_r = 4$

NVD Codes achieve the tradeoff (I)

Definition

Let $\mathscr{A}$ be an alphabet that is scalably dense, i.e.,

$$|\mathscr{A}(\mathsf{SNR})| \doteq \mathsf{SNR} \text{ and } a \in \mathscr{A}(\mathsf{SNR}) \Rightarrow |a|^2 \dot{\leq} \mathsf{SNR}.$$

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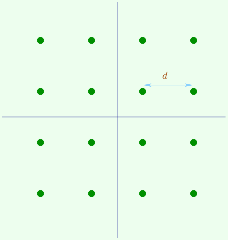
Then, a $n_t \times n_t$ space-time code χ is called a rate-n NVD code if it

1. is $\mathscr{A}$-linear
2. transmits on average n symbols **p**er **c**hannel **u**se (pcu) from the signal constellation $\mathscr{A}$.
3. has the non-vanishing determinant (NVD)[a] property.

[a] That means: a minimum determinant independent from the size of the constellation.



NVD Codes achieve the tradeoff (II)

Theorem

For any linear block fading channel

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{X} + \mathbf{Z}$$

where $\mathbf{H}$ *is a* $n_r \times n_t$ *channel with* $q = \min\{n_r, n_t\}$ *and* $\mathbf{Z}$ *is the AWGN with i.i.d. entries, the achievable D-M tradeoff of a rate-n NVD code* χ *satisfies*

$$d_{\chi}(r) \geq d_{out}\left(\frac{q}{n} r\right) \tag{6}$$

where $d_{out}(r)$ *is the outage upper bound of the D-M tradeoff for the channel* $\mathbf{H}$.

Outline

The Alamouti code and the D-M tradeoff

Properties of the Alamouti code

A codeword is

$$\mathbf{X} = \begin{bmatrix} s_1 & s_2 \\ -s_2^\star & s_1^\star \end{bmatrix} \quad (7)$$

with $s_1 = p_1 + iq_1$ and $s_2 = p_2 + iq_2$ being the 2 information symbols.

1. The code is linear
2. It has a non vanishing determinant (minimum Euclidean norm $|s_1|^2 + |s_2|^2$)
3. It is a rate-1 NVD code (1 symbol per channel use)

- It achieves the tradeoff $d_{\text{out}}(2r) = (2-2r)^+ \cdot (n_r - 2r)^+$.

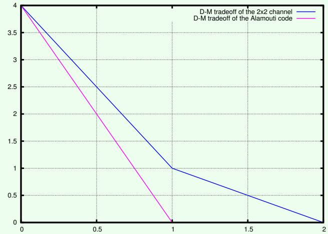


Figure: D-M tradeoff of the Alamouti code on the 2×2 channel

Cyclic division algebras (I)

- Let $\mathbb{F}$ be a field denoted the base field.
- Let $\mathbb{K}$ be a cyclic extension of degree n over $\mathbb{F}$. We define e such that $e^n = \gamma \in \mathbb{F}$ and

$$\forall z \in \mathbb{K}, z \cdot e = e \cdot \sigma(z)$$

  where σ is the generator of $\mathrm{Gal}_{\mathbb{K}/\mathbb{F}}(\mathbb{K})$. Then a cyclic algebra is

$$\mathscr{A} = (\mathbb{K}/\mathbb{F}, \sigma, \gamma) = \left\{ \sum_{i=0}^{n-1} z_i e^i \quad z_i \in \mathbb{K} \right\}.$$

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Matrix Formulation in dimension 3

$$\mathbf{X}_z = \begin{bmatrix} z & 0 & 0 \\ 0 & \sigma(z) & 0 \\ 0 & 0 & \sigma^2(z) \end{bmatrix} \qquad \mathbf{X}_e = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & 0 & 0 \end{bmatrix} \tag{8}$$

$$\mathbf{X} = \begin{bmatrix} z_0 & z_1 & z_2 \\ \gamma\sigma(z_2) & \sigma(z_0) & \sigma(z_1) \\ \gamma\sigma^2(z_1) & \gamma\sigma^2(z_2) & \sigma^2(z_0) \end{bmatrix}$$

Cyclic division algebras (II)

Definition

A cyclic algebra is a division algebra (CDA) *iff* each non zero element has an inverse. A cyclic division algebra is also named "skew field".

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Theorem

If

$$\gamma, \gamma^2, \ldots, \gamma^{n-1} \notin N_{\mathbb{K}/\mathbb{Q}}(\mathbb{K})$$

then the algebra is a cyclic division algebra.

Alamouti code and CDAs

Alamouti

Take $\mathbb{F} = \mathbb{R}$, $\mathbb{K} = \mathbb{C}$, $\gamma = -1$. $\sigma : x \mapsto x^*$. The algebraic norm of x is $|x|^2$. -1 is not a norm. Finally, we get the cyclic division algebra

$$\mathscr{A} = (\mathbb{K}/\mathbb{F}, \sigma, \gamma) = \left\{ \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}, \quad x_1, x_2 \in \mathbb{C} \right\}$$

which corresponds to the Alamouti scheme (Hamilton quaternions). If $x_k \in \mathbb{Z}[i]$ (QAM), then minimum squared determinant is in $\mathbb{Z}^+ \setminus \{0\}$ (rate-1 NVD code). Alamouti code is tradeoff achieving for 1 receive antenna.

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2 receive antennas

- We would like in fact a rate-2 NVD code.
- **Idea:** Find another CDA on another base field.

Part III

From the Golden code to Perfect STBCs

Outline

The Golden algebra

- Take as a base field $\mathbb{F} = \mathbb{Q}(i)$. The extension of degree 2 over $\mathbb{F}$ is $\mathbb{Q}\left(i, \sqrt{5}\right)$. $\gamma = i \in \mathbb{Z}[i]$.

The Golden algebra

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Matrix representation

$$\begin{pmatrix} a_1 + a_2\sqrt{5} & a_3 + a_4\sqrt{5} \\ i\left(a_3 - a_4\sqrt{5}\right) & a_1 - a_2\sqrt{5} \end{pmatrix}$$

with $a_k \in \mathbb{F} = \mathbb{Q}(i)$.

The Golden Code [Belfiore Rekaya Viterbo 04/05]

Rate-2 NVD Code

Codewords of the Golden Code are

$$\mathbf{X} = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha\,(z_1 + z_2\theta) & \alpha\,(z_3 + z_4\theta) \\ i \cdot \bar{\alpha}\,(z_3 + z_4\bar{\theta}) & \bar{\alpha}\,(z_1 + z_2\bar{\theta}) \end{pmatrix} \tag{9}$$

with $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + i - i\theta$, $\bar{\alpha} = 1 + i - i\bar{\theta}$ and $z_j \in \mathbb{Z}[i]$.

The Golden Code [Belfiore Rekaya Viterbo 04/05]

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with $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + i - i\theta$, $\bar{\alpha} = 1 + i - i\bar{\theta}$ and $z_j \in \mathbb{Z}[i]$.

- We can also write (9) in a split form,

$$\mathrm{vec}\mathbf{X} = \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha & \alpha\theta & 0 & 0 \\ 0 & 0 & \alpha & \alpha\theta \\ 0 & 0 & i\bar{\alpha} & i\bar{\alpha}\bar{\theta} \\ \bar{\alpha} & \bar{\alpha}\bar{\theta} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \quad (10)$$

  - Equation (10) is useful for the decoder. α has been chosen such that matrix of eq. (10) is unitary.

Minimum determinant

- The determinant of **X** is

$$
\begin{aligned}
\det \mathbf{X} &= \frac{1}{5}\left[N_{\mathbb{K}/\mathbb{Q}(i)}\left(\alpha\left(z_1+z_2\theta\right)\right) - iN_{\mathbb{K}/\mathbb{Q}(i)}\left(\alpha\left(z_3+z_4\theta\right)\right)\right] \\
&= \frac{2+i}{5}\left[N_{\mathbb{K}/\mathbb{Q}(i)}\left(z_1+z_2\theta\right) - iN_{\mathbb{K}/\mathbb{Q}(i)}\left(z_3+z_4\theta\right)\right]
\end{aligned}
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$$= \frac{2+i}{5}\left[N_{\mathbb{K}/\mathbb{Q}(i)}\left(z_1+z_2\theta\right) - iN_{\mathbb{K}/\mathbb{Q}(i)}\left(z_3+z_4\theta\right)\right]$$

- Since i is not a norm (*next slide*) of an element of $\mathbb{Q}(i)$, then

$$N_{\mathbb{K}/\mathbb{Q}(i)}\left(z_1+z_2\theta\right) - iN_{\mathbb{K}/\mathbb{Q}(i)}\left(z_3+z_4\theta\right) \quad \begin{matrix} \neq 0 \\ \in \mathbb{Z}[i] \end{matrix}$$

So,

$$\left|N_{\mathbb{K}/\mathbb{Q}(i)}\left(z_1+z_2\theta\right) - iN_{\mathbb{K}/\mathbb{Q}(i)}\left(z_3+z_4\theta\right)\right|^2 \geq 1$$

Minimum determinant

- The determinant of $\mathbf{X}$ is

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$$= \frac{2+i}{5}\left[N_{\mathbb{K}/\mathbb{Q}(i)}\left(z_1+z_2\theta\right) - iN_{\mathbb{K}/\mathbb{Q}(i)}\left(z_3+z_4\theta\right)\right]$$

- Since i is not a norm (*next slide*) of an element of $\mathbb{Q}(i)$, then

$$N_{\mathbb{K}/\mathbb{Q}(i)}\left(z_1+z_2\theta\right) - iN_{\mathbb{K}/\mathbb{Q}(i)}\left(z_3+z_4\theta\right) \quad \begin{array}{c}\neq 0 \\ \in \mathbb{Z}[i]\end{array}$$

So,

$$\left|N_{\mathbb{K}/\mathbb{Q}(i)}\left(z_1+z_2\theta\right) - iN_{\mathbb{K}/\mathbb{Q}(i)}\left(z_3+z_4\theta\right)\right|^2 \geq 1$$

Result

We get

$$\delta_{\min} = \min_{\mathbf{X}\neq 0} \left|\det \mathbf{X}\right|^2 = \left|\frac{2+i}{5}\right|^2 = \frac{1}{5}$$

i is not a norm

Sketch of proof [Oggier 04]

$\mathbb{Q}(i)$ can be embedded in $\mathbf{Q}_5$ by

$$i \mapsto 2 + 5\mathbf{Z}_5$$

- If i was a norm, then $\exists a + b\sqrt{5} \in \mathbb{Q}(i)$ such that $a^2 - 5b^2 = i$ which can be lifted in $\mathbf{Q}_5$ as $a^2 - 5b^2 = 2 + 5x, x \in \mathbf{Z}_5$.

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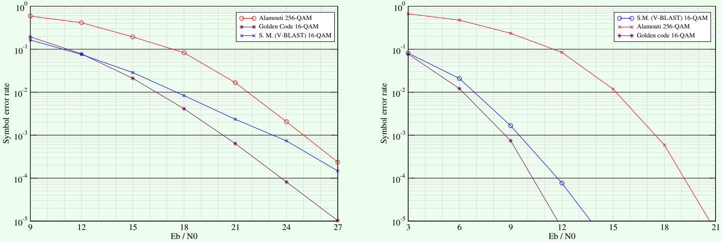
- It implies

$$v_5\left(a^2 - 5b^2\right) = \min\{2v_5(a), 2v_5(b) + 1\} = 0$$

so $a, b \in \mathbf{Z}_5$. Reducing equality mod $5\mathbf{Z}_5$, we find that 2 should be a square in $\mathbb{F}_5$.

Numerical results

Golden Code *vs* Alamouti and S.M.



(a) 2 receive antennas　　(b) 6 receive antennas

Figure: Numerical results 8 bits p.c.u.

Outline

Shaping

- Each layer of the code designed from cyclic division algebras is composed of a vector whose expression is $\left(z, \sigma(z), \ldots, \sigma^{n-1}(z)\right)$ (see (8)). z linearly depends on n information QAM or HEX symbols x_i. We have

$$\begin{pmatrix} z \\ \sigma(z) \\ \vdots \\ \sigma^{n-1}(z) \end{pmatrix} = \mathbf{G} \cdot \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

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- We impose, in order to preserve the mutual information of the MIMO channel, that $\mathbf{G}$ is a unitary matrix up to a normalization factor. This can usually be done by imposing that codewords components are in an ideal of $\mathscr{O}_{\mathbb{K}}$.
  - Moreover, we also impose that $|\gamma| = 1$ to balance the average energy over all the layers.

Definition

Space-Time Block codes associated to a matrix $\mathbf{G}$ which is a similitude and such that $|\gamma| = 1$ are said to have the "unitarity" property.

Perfect STBCs [Oggier *et al.* 04/06]

Definition

Perfect space-time block codes are codes constructed from a cyclic division algebra on $\mathbb{F} = \mathbb{Q}(i)$ (resp. $\mathbb{F} = \mathbb{Q}(j)$) by using QAM (resp. HEX) symbols that have the unitarity property.

Corollary

Perfect STBCs are tradeoff achieving and information preserving (optimum behavior at high and low signal to noise ratios).

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Example: The Golden Code

The Golden Code is a perfect space-time block code. The ideal of $\mathcal{O}_{\mathbb{K}}$ is a principal ideal generated by α.

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Theorem

Perfect space-time block codes are rate-n_t NVD codes.

3, 4 and 6 antennas

3 antennas: the parameters

- $\mathbb{F} = \mathbb{Q}(j)$, $\mathbb{K} = \mathbb{Q}(j,\theta)$ with $\theta = 2\cos\left(\frac{2\pi}{7}\right)$
- $\gamma = j$, $\alpha = ((1+j)+\theta)$
- $\delta_{\min}(\mathscr{C}_\infty) = \frac{1}{49}$

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4 antennas: the parameters

- $\mathbb{F} = \mathbb{Q}(i)$, $\mathbb{K} = \mathbb{Q}(i, \theta)$ with $\theta = 2\cos\left(\frac{2\pi}{15}\right)$
- $\gamma = i$, $\alpha = (1 - 3i) + i\theta^2)$
- $\delta_{\min}(\mathscr{C}_\infty) = \frac{1}{1125}$

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6 antennas: the parameters

- $\mathbb{F} = \mathbb{Q}(j)$, $\mathbb{K} = \mathbb{Q}(j, \theta)$ with $\theta = 2\cos\left(\frac{\pi}{14}\right)$
- $\gamma = -j$, non principal ideal
- $\frac{1}{2^6.7^5} \leq \delta_{\min}(\mathscr{C}_\infty) \leq \frac{1}{2^6.7^4}$

Other perfect STBCs

- [Elia et al. 05] found perfect STBCs for any number of antennas by relaxing the constraint on γ.
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Hint

Take $\gamma = \frac{a \pm ib}{b \pm ia}$ $a, b \in \mathbb{Z}$ where $|a + ib|^2$ is not a norm of $\mathbb{K}/\mathbb{Q}(i)$.
To find non norm elements is quite easy. Take $p \equiv 1\ (4)$ such that $p\mathcal{O}_{\mathbb{K}}$ is a prime ideal. Decompose $p = a^2 + b^2$.

Part IV

Further applications

Outline

9 MIMO OFDM

10 Decoding

11 Encoding the Golden Code

System Model: An example

- Parallel MIMO channel: Consider the case of a MIMO OFDM system
  - On each subcarrier, the channel is considered flat
  - When two such subcarriers f_i and f_j are separated by at least the channel coherence bandwidth B_c, then channel coefficients on these subcarriers are decorrelated.

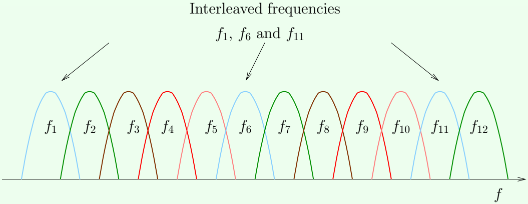

Figure: OFDM as parallel channels: here, channel coefficients on frequencies f_1, f_6, and f_{11} are considered decorrelated

System Model: The equivalent channel

- Denote n_t the number of transmit antennas, n_r the number of receive antennas and N the number of parallel channels

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Received signal

$$\boxed{\mathbf{Y}_{Nn_r \times T} = \mathbf{H}_{Nn_r \times Nn_t} \cdot \mathbf{X}_{Nn_t \times T} + \mathbf{Z}_{Nn_r \times T}}$$

where **X** is the transmitted codeword, **Z** is the i.i.d. noise and **H** is the channel matrix.

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where $\mathbf{X}$ is the transmitted codeword, $\mathbf{Z}$ is the i.i.d. noise and $\mathbf{H}$ is the channel matrix.

- More precisely, we have

$$\mathbf{H} = \mathrm{diag}\left(\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_N\right)$$

where $\mathscr{H}_i$ is a $n_r \times n_t$ submatrix corresponding to channel i (for example the interleaved subcarrier f_i in an OFDM system).

Block-Diagonal STBCs from division algebras

Idea

Let the base field be $\mathbb{F}$, an extension of degree N over $\mathbb{Q}(i)$ (resp. $\mathbb{Q}(j)$) with Galois group

$$\mathrm{Gal}\left(\mathbb{F}/\mathbb{Q}(i)\right) = \{\tau_1, \tau_2, \ldots, \tau_N\}.$$

$\mathbb{K}$ is a cyclic extension of degree n_t of $\mathbb{F}$. Codewords have the form

$$\mathbf{X} = \begin{bmatrix} \boxed{\tau_1(\Xi)} & & & \\ & \boxed{\tau_2(\Xi)} & & \\ & & \ddots & \\ & & & \boxed{\tau_N(\Xi)} \end{bmatrix}$$

where Ξ is the matrix representation of an element of a cyclic division algebra with base field $\mathbb{F}$ and maximal subfield $\mathbb{K}$.

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Non Vanishing Determinant

Minimum squared determinant is

$$\boxed{\det \mathbf{X} = \prod_{i=1}^{N} \det \tau_i(\Xi) = \prod_{i=1}^{N} \tau_i(\det \Xi) = N_{\mathbb{F}/\mathbb{Q}(i)}(\det \Xi) \in \mathbb{Z}[i] \setminus \{0\}}$$

Perfect Codes MIMO OFDM: An example

- Consider the case $n_t = n_r = 2$ and $N = 2$ parallel channels

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2-Golden Code

Codeword is

$$\mathbf{X} = \begin{bmatrix} \Xi & \mathbf{0} \\ \mathbf{0} & \tau(\Xi) \end{bmatrix}$$

where

$$\Xi = \begin{bmatrix} \alpha\cdot(s_1 + s_2\zeta_8 + s_3\theta + s_4\zeta_8\theta) & \alpha\cdot(s_5 + s_6\zeta_8 + s_7\theta + s_8\zeta_8\theta) \\ \zeta_8\bar{\alpha}\cdot\left(s_5 + s_6\zeta_8 + s_7\bar{\theta} + s_8\zeta_8\bar{\theta}\right) & \bar{\alpha}\cdot\left(s_1 + s_2\zeta_8 + s_3\bar{\theta} + s_4\zeta_8\bar{\theta}\right) \end{bmatrix}$$

with $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + i - i\theta$, $\bar{\alpha} = 1 + i - i\bar{\theta}$, $\tau : \zeta_8 \mapsto -\zeta_8$ and $s_k \in \mathbb{Z}[i]$.

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Proposition

The 2−Golden Code is a perfect code for the $(2,2,2)$ parallel MIMO channel

Outline

Reduction-aided decoding

- Infinite code formulation

$$\mathbf{Y}_{n_t\times n_t} = \mathbf{H}_{n_t\times n_t}\cdot\mathbf{X}_{n_t\times n_t} + \mathbf{W}_{n_t\times n_t}$$

where **X** is a codeword derived from a cyclic division algebra.

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  where $\mathbf{X}$ is a codeword derived from a cyclic division algebra.

- Let $\mathbf{U}$ be a unit in the cyclic division algebra that approximate

$$\left(\frac{1}{|\det \mathbf{H}|}\right)^{\frac{1}{n_t}} \cdot \mathbf{H}$$

  Denote $\mathbf{E} = \mathbf{H} \cdot \mathbf{U}^{-1}$. Then, $\mathbf{Z} = \mathbf{U} \cdot \mathbf{X}$ remains in the code and the received signal becomes

$$\mathbf{Y} = \mathbf{E} \cdot \mathbf{Z} + \mathbf{W}$$

  up to a unitary matrix, with matrix $\mathbf{E}$ better conditioned for suboptimal decoding algorithms.

Outline

Binary labelling

- The encoder has binary symbols at its input and must label codewords of the Golden codes with these bits.

Ungerboeck-type labelling

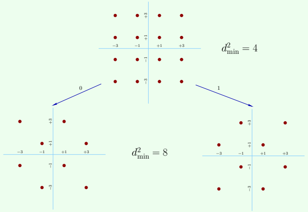

Figure: Ungerboeck labelling of a QAM constellation

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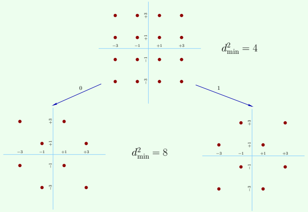

Figure: Ungerboeck labelling of a QAM constellation

- We propose here a Ungerboeck-type binary labelling for the Golden code that has the minimum determinant as design criterion.

Partition chain (I)

- Let β be an element of the Golden algebra $\beta = i\,(1-\theta) + (1-\theta)\,e$ with matrix representation

$$\mathbf{X}_\beta = \begin{bmatrix} i\,(1-\theta) & 1-\theta \\ i\left(1-\bar{\theta}\right) & i\left(1-\bar{\theta}\right) \end{bmatrix}$$

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Definition

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- Because $N_r(\beta) = 1+i$, the Golden code can be partitioned into 4 subsets: $\mathfrak{P}$ and 3 translated versions.
- Minimum determinant of each subset is $\boxed{\delta_{\text{min}} = 2 \cdot \delta_{\text{min}}(\mathscr{G})}$

Partition chain (II)

Lattice and ideal partition

$$
\begin{array}{ccc}
\mathbb{Z}[i]^4 \cong \mathbb{Z}^8 & \longleftrightarrow & \mathcal{G} \\
\downarrow & & \downarrow \\
D_4^2 & \longleftrightarrow & \mathfrak{P} \\
\downarrow & & \downarrow \\
E_8 & \longleftrightarrow & \mathfrak{P}^2 \\
\downarrow & & \downarrow \\
L_8 & \longleftrightarrow & (1+i)\mathfrak{P} \\
\downarrow & & \downarrow \\
2\mathbb{Z}^8 & \longleftrightarrow & 2\mathcal{G}
\end{array}
$$

Labelling

- Non-$\mathbb{Z}^8$ lattices based constellations to label (E_8 for example).

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Figure: Details in a next talk

Thank you for your attention!