Wrights' Axioms of Queuing Theory

1. If you have a choice between waiting here and waiting there, wait there.
2. All things being equal, it is better to wait at the front of the line.
3. There is no point in waiting at the end of the line. But note Wrights' Paradox:
4. If you don't wait at the end of the line, you'll never get to the front.
5. Whichever line you are in, the others always move faster.

Queueing Theory Analysis of the Provincial Acute Healthcare System

Alexander Rutherford

sandyr@irmacs.sfu.ca

Complex Systems Modelling Group

Azadeh Alimadad (Health Sci. & IRMACS)
Afsaneh Bakhtiari (Math. & IRMACS)
Alexandra Borwein (Undergrad. RA)
Peter Borwein (Math. & IRMACS)
Vahid Dabbaghian (Math. & IRMACS)
Aditya Damani (Undergrad. RA)
Hannah Dodd (Undergrad. RA)
Chiaka Drakes (Math)
Ron Ferguson (Math. & IRMACS)
Ellen Fowler (IRMACS)
Amir Ghaseminejad (Comm. & IRMACS)
Christopher Giles (Crim. & IRMACS)

Yuri Gusev (IRMACS)
Warren Hare (Math. & IRMACS)
Michel Joffres (Health Sci.)
Jenny Li (Math. & IRMACS)
Viviane Dias Lima (BCCE in HIV)
Natasha Richardson (Undergrad. RA)
Alexander Rutherford (Math. & IRMACS)
Alexa van der Waall (Math. & IRMACS)
Krisztina Vasarhelyi (IRMACS)
Les Vertesi (Health Sci. & IRMACS)
Yanchao Wang (IRMACS)
J. F. Williams (Math.)

Current Projects

1. Surgical Wait List Modelling
2. Acute Health Care System
3. Home and Community Health Care System
4. Criminal Justice System
5. Epidemiology of HIV
6. Determinants of Obesity
7. Computational Criminology
8. Risk Analysis for Fires

Modelling Access to Acute Care

How do we measure access to acute care?

Project Team

Chiaka Drakes
Ron Ferguson
Warren Hare
Jenny Li
A. R.
Krisztina Vasarhelyi
Les Vertesi
Yanchao Wang

Measuring Access to Acute Care

Target Occupancy Ratio (TOR)

The average proportion of inpatient beds in the hospital that are occupied.

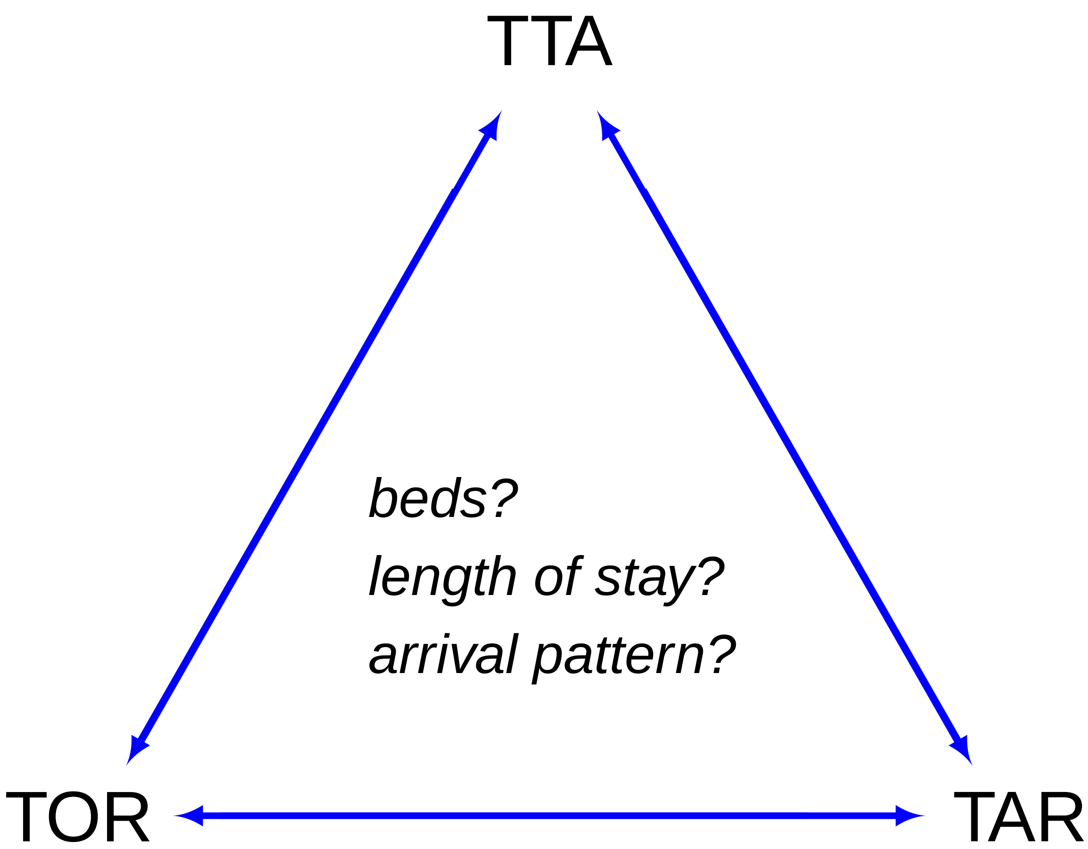
Target Access Ratio (TAR)

The fraction of time that the hospital has enough inpatient beds for all admitted patients.

Target Time to Access (TTA)

The expected wait time for a percentile of patients to receive an inpatient bed. For example, 80% of admitted patients receive an inpatient bed within 6 hours.

How are the Access Metrics Related?



Queueing Theory

Agner Krarup Erlang

Jan. 1, 1878	Born at Lønber in Jutland, Denmark
1892	Passed the Preliminary Examinations to the University of Copenhagen at 14 years old.
1896	Passed Entrance Exams to the University of Copenhagen
1891	Graduated with an MA in Mathematics
1908	Joined the Danish Telephone Company
1909	Publish *The Theory of Probabilities and Telephone Conversations*
1917	Published *Solutions of Some Problems in the Theory of Probabilities of Significance in Automatic Telephone Exchanges*
Feb. 3, 1929	Died following an abdominal operation.

The Theory of Probabilities and Telephone Conversations

1. THE THEORY OF PROBABILITIES AND TELEPHONE CONVERSATIONS

First published in "Nyt Tidsskrift for Matematik" B, Vol. 20 (1909), p. 33.

Although several points within the field of Telephony give rise to problems, the solution of which belongs under the Theory of Probabilities, the latter has not been utilized much in this domain, so far as can be seen. In this respect the Telephone Company of Copenhagen constitutes an exception as its managing director, Mr. *F. Johannsen*, through several years has applied the methods of the theory of probabilities to the solution of various problems of practical importance; also, he has incited others to work on investigations of similar character. As it is my belief that some point or other from this work may be of interest, and as a special knowledge of telephonic problems is not at all necessary for the understanding thereof, I shall give an account of it below.

The Standard Model for a Hospital

$M/M/b$ Queue

Chapman-Kolmogorov Equations

Let λ by the arrival rate and μ^{-1} the average length of stay. Then, $p_n(t)$, the probability that there are n patients in the system at time t satisfies

$$\frac{dp_0}{dt} = -\lambda p_0 + \mu\, p_1$$

$$\frac{dp_n}{dt} = -(\lambda + n\mu)p_n + (n+1)\mu p_{n+1} + \lambda p_{n-1}\,, \qquad \text{for } 1 \leq n \leq b-1$$

$$\frac{dp_n}{dt} = -(\lambda + b\mu)p_n + b\mu p_{n+1} + \lambda p_{n-1}\,, \qquad \text{for } b \geq b$$

Chapman-Kolmogorov Equations

At stochastic equilibrium,

$$0 = \frac{dp_0}{dt} = -\lambda p_0 + \mu\, p_1$$

$$0 = \frac{dp_n}{dt} = -(\lambda + n\mu)p_n + (n+1)\mu p_{n+1} + \lambda p_{n-1}\,, \qquad \text{for } 1 \leq n \leq b-1$$

$$0 = \frac{dp_n}{dt} = -(\lambda + b\mu)p_n + b\mu p_{n+1} + \lambda p_{n-1}\,, \qquad \text{for } b \geq b$$

Therefore, if $\lambda < b\mu$

$$p_n = \begin{cases} \frac{a^n}{n!}\, p_0\,, & \text{for } 1 \leq n \leq b-1 \\ \frac{a^n}{b!\, b^{n-b}}\, p_0\,, & \text{for } n \geq b \end{cases}$$

where

$$p_0 = \frac{(b-1)!(b-a)}{e^a(b-a)\Gamma(b,a) + a^b} \qquad \text{and} \qquad a = \frac{\lambda}{\mu}$$

TOR and TAR for an $M/M/b$ Queue

$$\text{TAR} = 1 - \frac{\rho}{e^a a^{-b}(b-a)\Gamma(b,a) + 1}$$

$$\begin{aligned}\text{TOR} &= \frac{\rho}{(e^a a^{-b}(b-a)\Gamma(b,a) + 1)b(1-\rho)} \\ &= \rho + \frac{1 - \text{TAR}}{b(1-\rho)}\end{aligned}$$

where $\rho = \frac{\lambda}{b\mu}$.

Wait Time Distribution for $M/M/b$

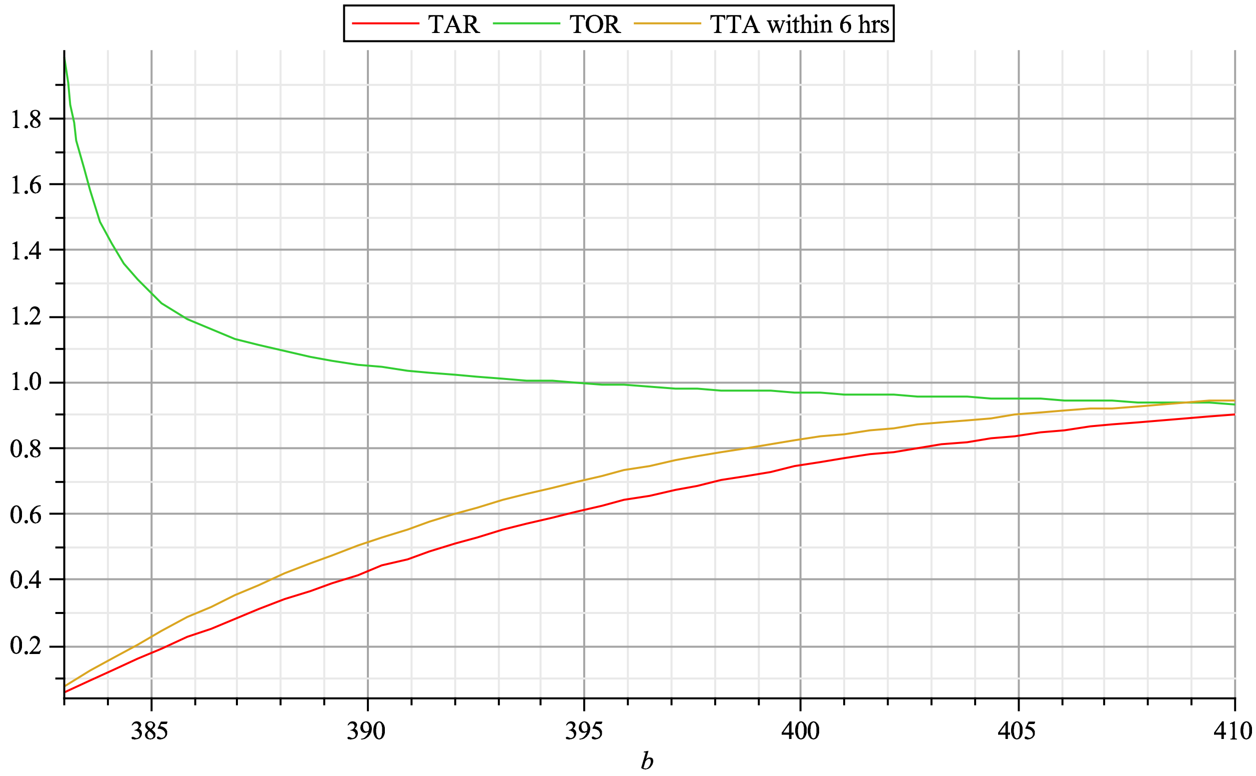
The probability of waiting time less than or equal to t is

$$W(t) = 1 - \frac{a^b e^{-\mu(b-a)t}}{e^a (b-a)\Gamma(b,a) + a^b}$$

The TTA for 6 hours is W(6).

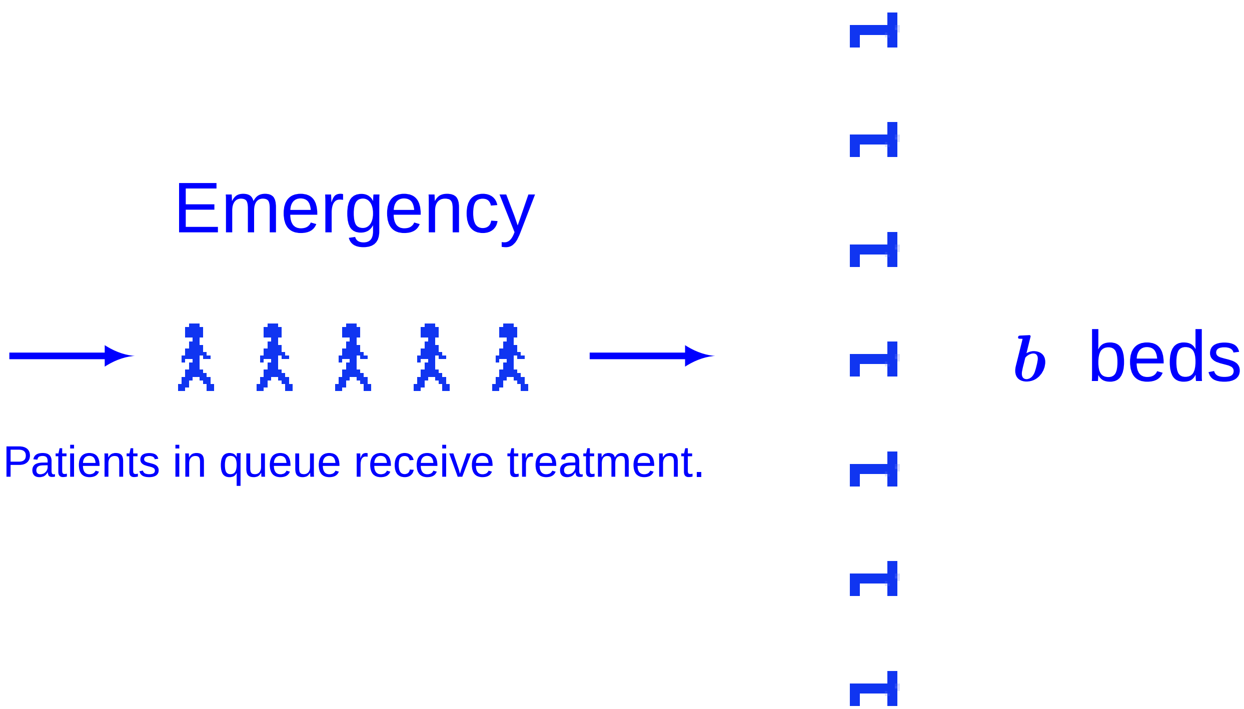
An Example Hospital

- Patients are admitted at a rate of 1.50 patients per hour.
- The average length of stay is 254.7 hours (10.6 days).
- Number of beds required for stochastic equilibrium is 381.2.



The Soft Queue Model

$M/M/\infty$ Queue with Beds



TOR and TAR for the Soft Queue

Stochastic equilibrium exists for all $\lambda > 0$ and $\mu > 0$.

$$p_n = \frac{e^{-a}a^n}{n!}$$

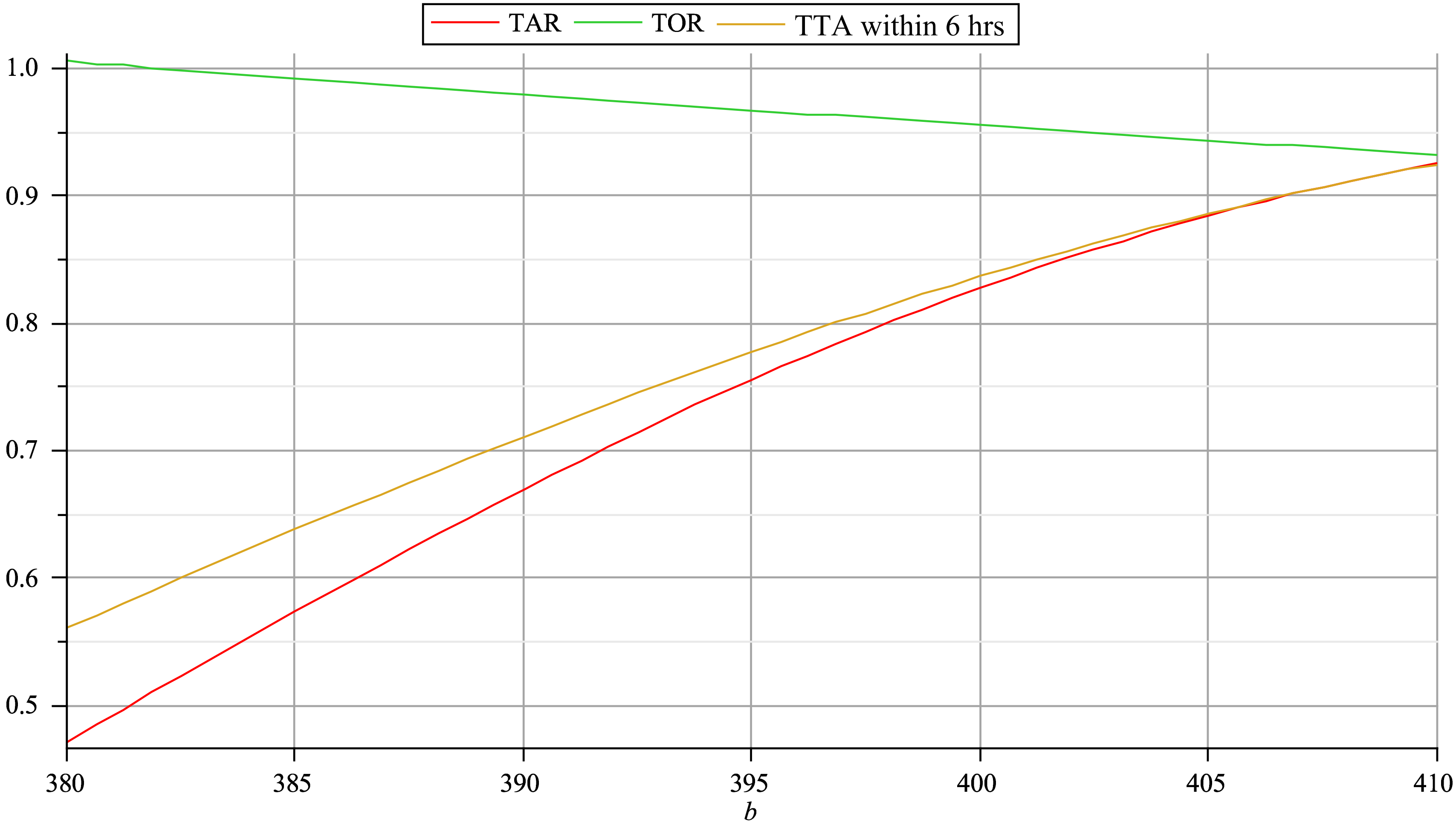
$$\text{TAR} = \frac{b\Gamma(b,a) + e^{-a}a^b}{b!}$$

$$\text{TOR} = \frac{a}{b}$$

Wait Time Distribution

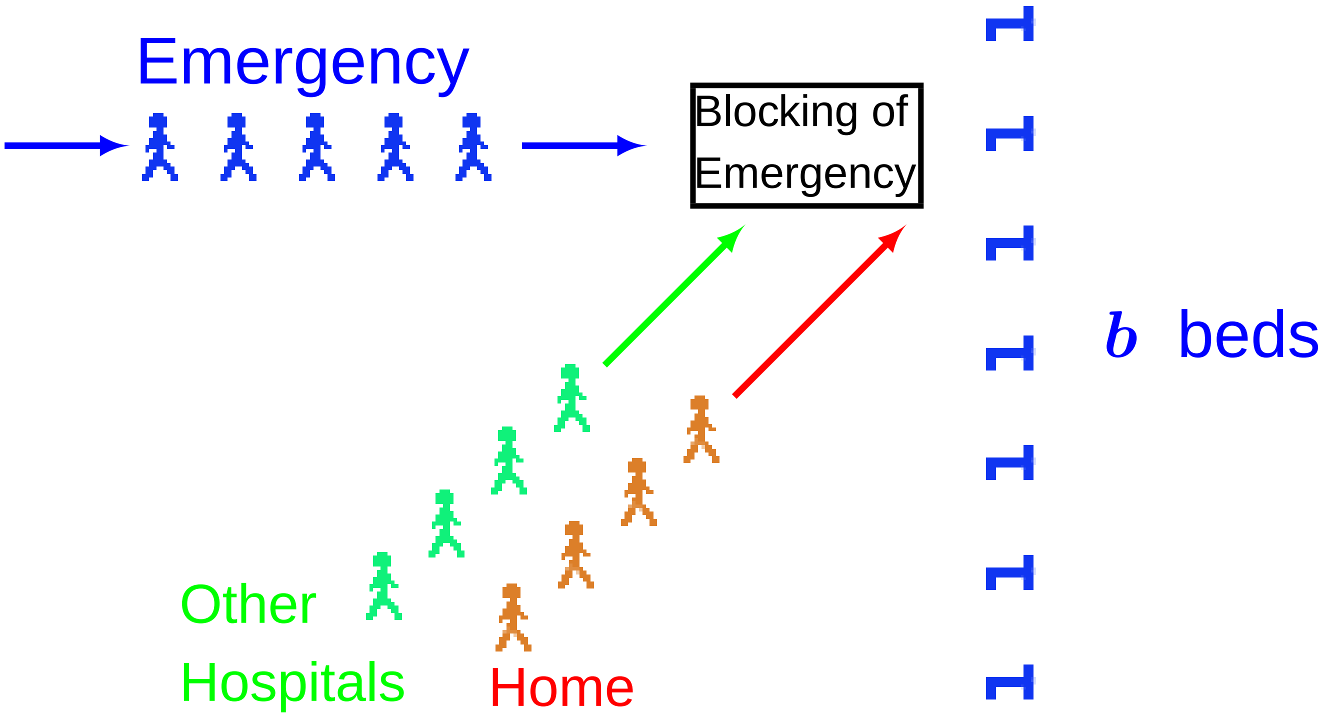
$$W(t) = 1 - e^{-\mu t}\left(1 - \frac{\Gamma(b,a)}{(b-1)!}\right)\left[1 - \frac{\Gamma(b, ae^{-\mu t}) - \Gamma(b,a)}{(b-1)!}\right]$$

The Example Hospital



Multistream Soft Queue

Nonstationary Arrival and LOS Distributions



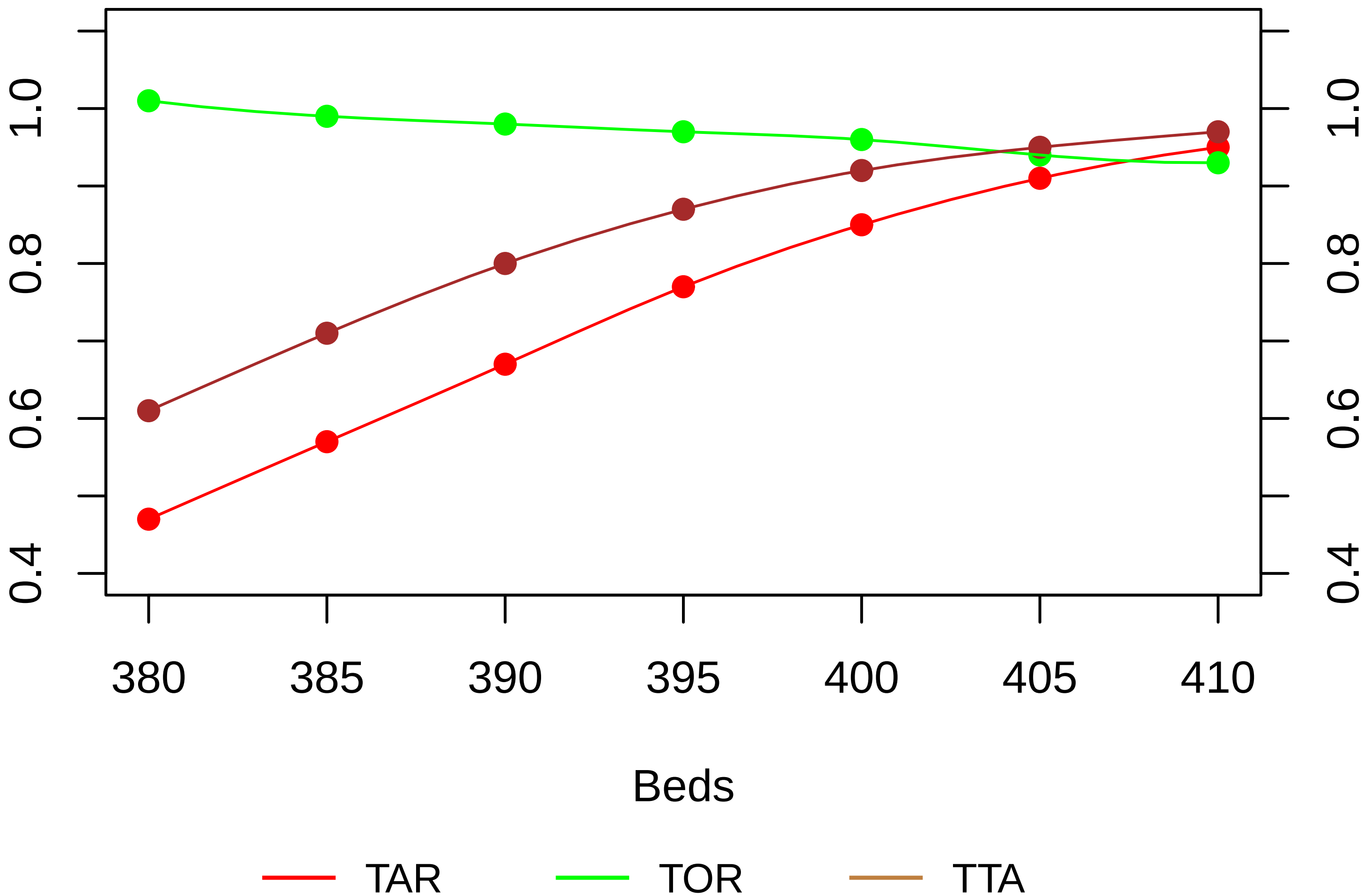
The Example Hospital

Stream E for fiscal 2005/06

	00-04	04-08	08-12	12-16	16-20	20-24	Total	Rate
Monday	126	125	122	237	345	384	**1339**	**0.017545**
Tuesday	190	135	115	256	321	361	**1378**	**0.018403**
Wednesday	183	123	121	235	340	357	**1359**	**0.018149**
Thursday	187	139	114	227	362	365	**1394**	**0.018616**
Friday	137	149	115	228	341	362	**1332**	**0.017788**
Saturday	151	122	123	216	270	298	**1180**	**0.015758**
Sunday	159	133	103	208	277	285	**1165**	**0.015558**
Total	**1133**	**926**	**813**	**1607**	**2256**	**2412**	**9147**	
Rate	**0.012934**	**0.010571**	**0.009281**	**0.018345**	**0.02575**	**0.027534**	**0.017403**	

Average LOS for stream E: 16588 minutes (11.5 days).

TOR, TAR, and TTA by Simulation



Next Steps

- Segmentation
- Connect hospitals together
- Demographic factors

The Moral of the Story

Mathematical modelling has a role to play as a lingua franca between different interests in developing healthcare policy.