Eigenvalues of $(3, 6)$-Fullerenes

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joint work with:

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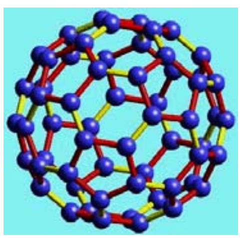
Coast-to-coast Seminar: 2007-09-25

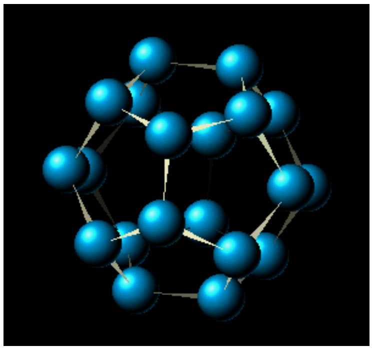
Richard Buckminster Fuller
1895 - 1983

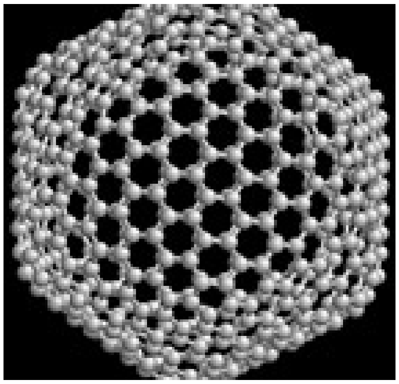


Buckyball - C_{60} molecule
1985

Definition: Fullerene

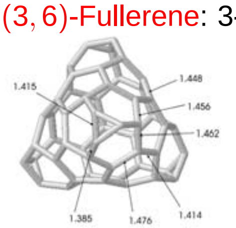
(5,6)-Fullerene: 3-regular, planar, (12) pentagons & hexagons

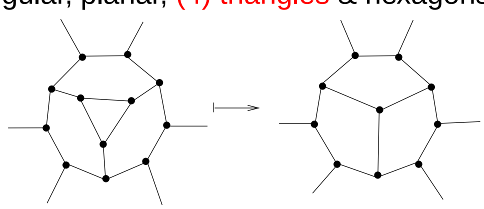




Definition: (3,6)-Fullerene

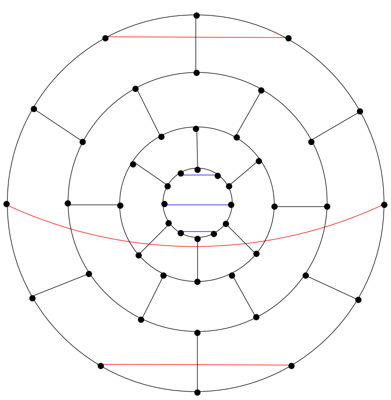
- $(3, 6)$-Fullerene: 3-regular, planar, (4) triangles & hexagons



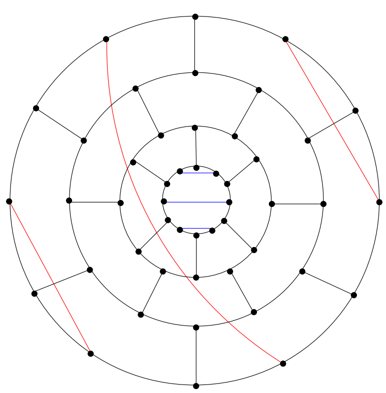


- Chemistry/Physics Interest:
  - They modify to “tetrahedral” (5,6)-Fullerenes
  - Can model “nanotubes”
  - Easier to characterize
  - **Critical Question:** Find “Energy Spectrum” = eigenvalues of adjacency matrix

Tube Construction of all (3,6)-Fullerenes



$(4, 6, 0) - \text{tube}$



$(4, 6, 1) - \text{tube}$

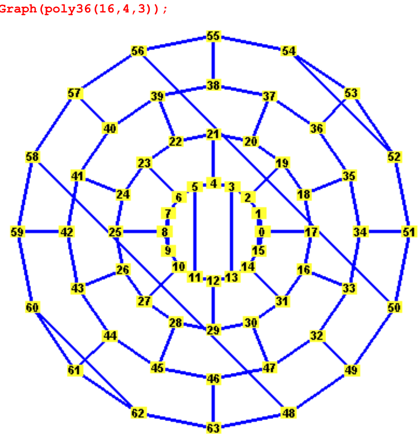
(length, girth, twist)

Specification is 3-to-1.

Maple Tests – GraphTheory package

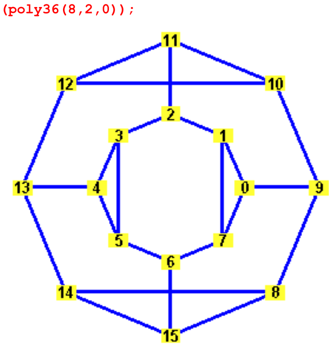
```
> cylinder := proc( n::posint, ell::posint )
  local v;
    if irem(n, 4) <> 0 then error "1st argument must be a multiple of 4";
  fi;
    v := (i,j) -> i*n+j;
    Graph( seq(Trail( seq(v(i,j), j=0..n-1), v(i,0) ), i=0..ell-1),
           { seq(seq({v(i,2*j), v(i+1,2*j+1)}, j=0..n/2-1), i=0..ell-2) }
  );
  end:
> poly36 := proc(n, ell, shift)
  local v, G;
    v := (i,j) -> i*n+j;
    G := AddEdge( cylinder(n,ell), { seq({v(0,2*i+1), v(0,n-(2*i+1))}, i=
  0..n/4-1), seq({v(ell-1,irem(2*shift+2*i,n)), v(ell-1,irem(2*shift+n-2*
  i-2, n))}, i=0..n/4-1) } );
    SetVertexPositions(G, [seq(seq(evalf((i+1)*[cos(2*Pi*(j-i)/n), sin(2*
  Pi*(j-i)/n)]), j=0..n-1), i=0..ell-1)]);
    G
  end:
```

```
> DrawGraph(poly36(16,4,3));
```



Maple Tests – Adjacency Matrix

```
> DrawGraph(poly36(8,2,0));
```




```
> AdjacencyMatrix(poly36(8,2,0));
```

$$\begin{bmatrix} 0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&1\\ 1&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0 \end{bmatrix} \tag{9}$$

Maple Tests – Spectrum

```
> Spectrum( poly36(8, 1, 0));
```

$$\left[-\sqrt{5}, -1, -1, -1, -1, 1, \sqrt{5}, 3\right] \quad (12)$$

```
> Spectrum( poly36(12, 1, 1));
```

$$\left[-\sqrt{7}, -\sqrt{3}, -1, -1, -1, -1, -1, 1, 1, \sqrt{3}, \sqrt{7}, 3\right] \quad (13)$$

```
> Spectrum( poly36(4, 4, 1));
```

$$\left[-\sqrt{5+2\sqrt{2}}, -\sqrt{5}, -\sqrt{5-2\sqrt{2}}, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1, \sqrt{5-2\sqrt{2}}, \sqrt{5}, \sqrt{5+2\sqrt{2}}, 3\right] \quad (14)$$

```
> Spectrum( poly36(12, 2, 0));
```

$$\left[-\sqrt{7}, -\sqrt{4+\sqrt{3}}, -\sqrt{5}, -\frac{1}{2}\sqrt{6}-\frac{1}{2}\sqrt{2}, -\sqrt{3}, -\sqrt{4-\sqrt{3}}, -1, -1, -1, -1, -1, -1, -1, -\frac{1}{2}\sqrt{6}+\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{6}-\frac{1}{2}\sqrt{2}, 1, 1, 1, \sqrt{4-\sqrt{3}}, \sqrt{3}, \frac{1}{2}\sqrt{6}+\frac{1}{2}\sqrt{2}, \sqrt{5}, \sqrt{4+\sqrt{3}}, \sqrt{7}, 3\right] \quad (15)$$

Fowler conjecture

Conjecture (Fowler, 1995)

The spectrum of every $(3, 6)$*-Fullerene takes the form*

$$\{3, -1, -1, -1\} \cup \Lambda \cup -\Lambda.$$

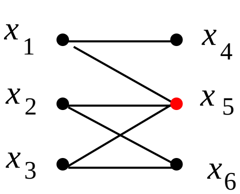
where Λ *is a multiset of reals.*

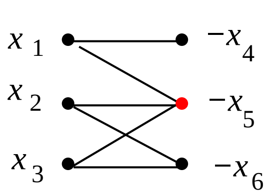
$(3, 6)$-Fullerenes are "spectrally almost bipartite"?

Lemma

G has spectrum $\Lambda \cup -\Lambda$ *iff G is bipartite.*

Eigenvector $x : V \to \mathbb{C}$ $\qquad \sum_{uv \in E} x(u) = \lambda x(v)$



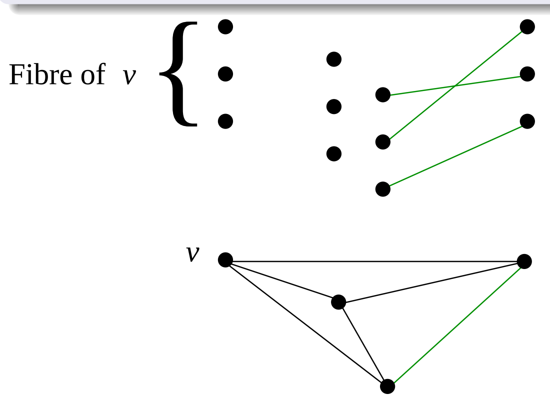


$x_1 + x_2 + x_3 = \lambda x_5$ $\qquad = (-\lambda)(-x_5)$

Why $\{3, -1, -1, -1\}$?

Lemma

Every $(3, 6)$*-Fullerene is a cover of* K_4.



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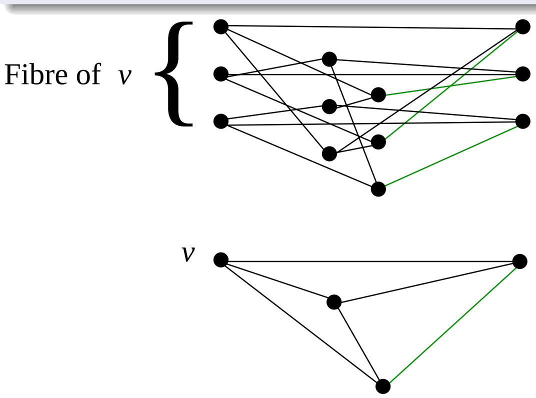
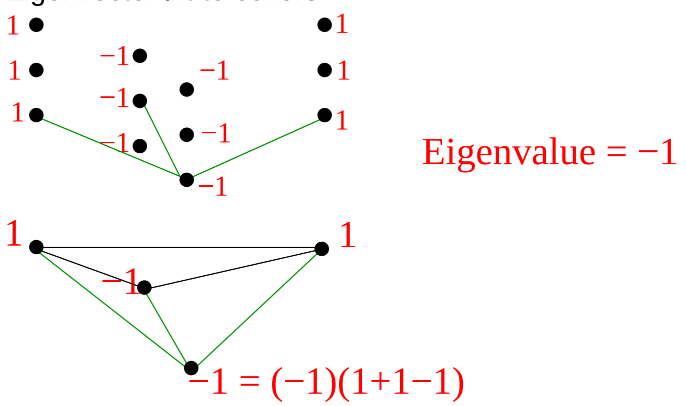
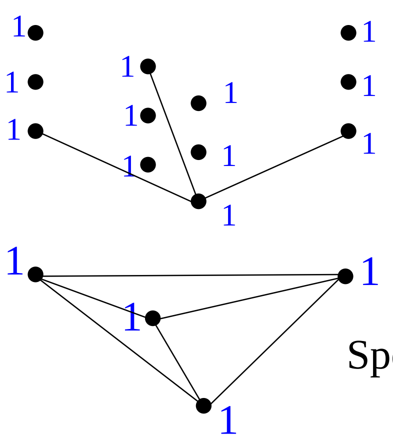


Why $\{3, -1, -1, -1\}$?

Eigenvector lift to covers



Why $\{3, -1, -1, -1\}$?



Eigenvalue = 3

$\mathrm{Spec}(K_4) = \{3, -1, -1, -1\}$

Experimenting. . .

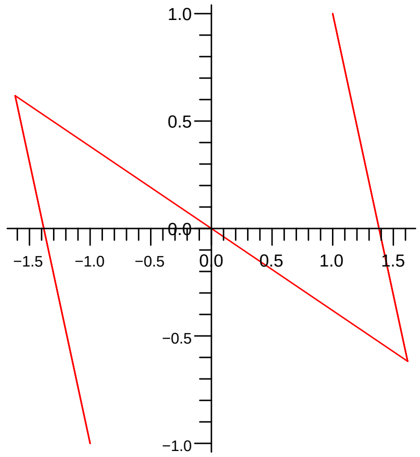
Idea

Where x, x' *are eigenvectors for* $\lambda, -\lambda$*,*
plot vertex v *at coordinates* (x_v, x'_v)*.*

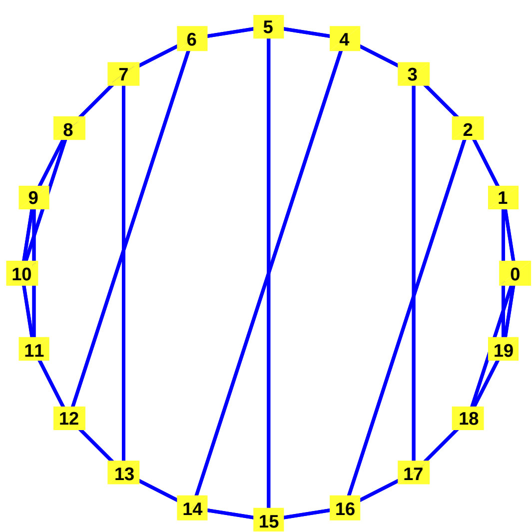
```
> lambda,X := Eigenvectors(AdjacencyMatrix(poly36(4, 2, 0))):
  Transpose(lambda);
```

$$\left[\begin{array}{cccccccc} 1 & \sqrt{5} & -\sqrt{5} & 3 & -1 & -1 & -1 & -1 \end{array}\right] \quad (15)$$

```
> plot(convert(SubMatrix(X,1..-1,[2,3]),listlist));
```



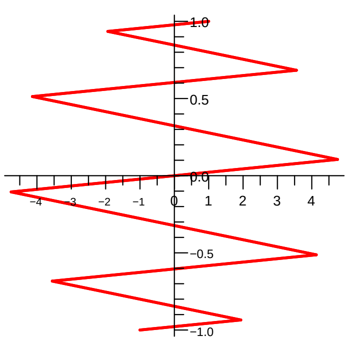
```
> G:=poly36(20,1,0): DrawGraph(G);
```



```
> lambda,X := Eigenvectors(AdjacencyMatrix(G)):
  Transpose(lambda);
```

$$\left[\left[-\sqrt{6+\sqrt{5}}, \sqrt{6+\sqrt{5}}, -\sqrt{6-\sqrt{5}}, \sqrt{6-\sqrt{5}}, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1, -\sqrt{4+\sqrt{5}}, \sqrt{4+\sqrt{5}}, -\sqrt{4-\sqrt{5}}, \sqrt{4-\sqrt{5}}, 3\right]\right] \quad (17)$$

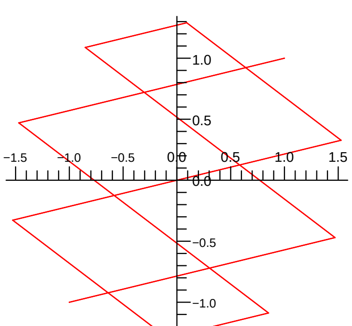
```
> plot(convert(SubMatrix(X,1..-1,[1,2]),listlist));
```



```
> lambda,X := Eigenvectors(AdjacencyMatrix(G)):
  Transpose(lambda);
```

$$\left[-\sqrt{6+\sqrt{5}}, \sqrt{6+\sqrt{5}}, -\sqrt{6-\sqrt{5}}, \sqrt{6-\sqrt{5}}, 3, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -\sqrt{4+\sqrt{5}}, \sqrt{4+\sqrt{5}}, -\sqrt{4-\sqrt{5}}, \sqrt{4-\sqrt{5}}\right] \quad (15)$$

```
> plot(convert(SubMatrix(X,1..-1,[3,4]),listlist));
```



```
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```

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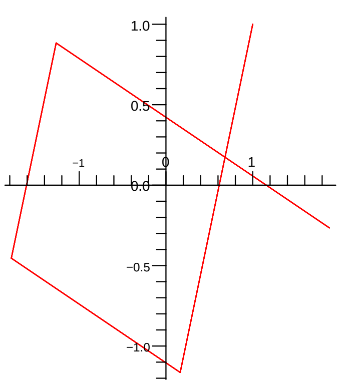
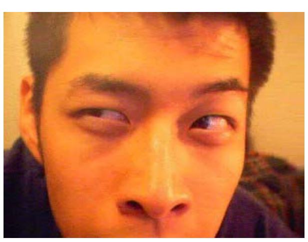
```
> plot(convert(SubMatrix(X,1..-1,[-1,-2]),listlist));
```





Suspicion

Where x, x' are (appropriately scaled) eigenvectors for $\lambda, -\lambda$, the complex numbers $x(v) + ix'(v)$ are (phase shifted) roots of unity.

A Similar Phenomenon

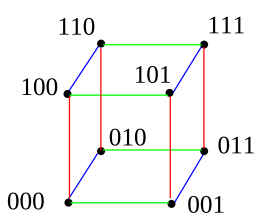
Fact

Abelian Cayley graphs have complex-root-of-unity eigenvectors (not phase shifted).

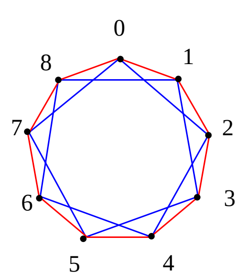
Cayley Graph

Γ = additive abelian group. $S \subseteq \Gamma$, the *symbol set*. $S = -S$

$\mathrm{Cay}(\Gamma, S)$: $V = \Gamma$, $E = \{uv \mid u - v \in S\}$



$\mathrm{Cay}(Z_2^3, \{100, 010, 001\})$



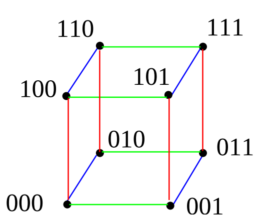
$\mathrm{Cay}(Z_9, \{\pm 1, \pm 2\})$

Cayley Sum Graph

Γ = additive abelian group $\quad S \subseteq \Gamma$, the *symbol set*. ~~$S = -S$~~

$\mathrm{Cay}(\Gamma, S)$: $\quad V = \Gamma \quad E = \{uv \mid u - v \in S\}$

CaySum(Γ, S) : $V = \Gamma \quad E = \{uv \mid u + v \in S\}$



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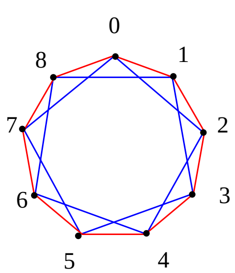
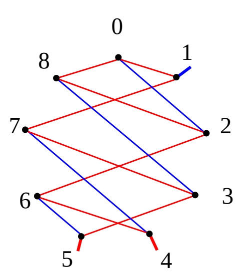
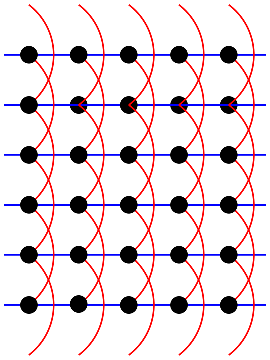


$\mathrm{Cay}(Z_9, \{\pm 1, \pm 2\})$

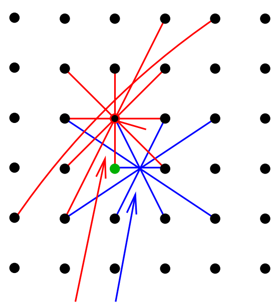


$\mathrm{CaySum}(Z_9, \{\pm 1, 2\})$

Geometric Cayley Sum Graphs



$Cay(\mathbb{Z}^2, \{(\pm 1, 0), (0, \pm 2)\})$



$CaySum(\mathbb{Z}^2, \{(1, 0), (0, 2)\})$

Cayley Sum Graphs - Facts

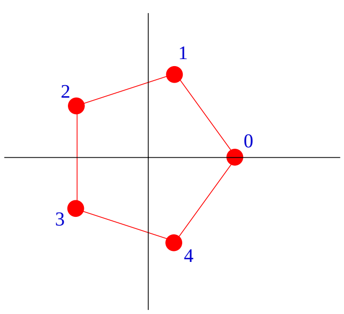
- $\mathrm{CaySum}(\Gamma, S)$ is $|S|$-regular (counting loops once)
- $\mathrm{CaySum}(\Gamma, S) \simeq \mathrm{CaySum}(\Gamma, S + 2t)$ (via $x \mapsto x + t$)
- $\mathrm{CaySum}(\Gamma, S)^{(2)} = \mathrm{Cay}(\Gamma, S - S)$
- (F. Chung, 1989) Good expansion
- (B. Green, 2003) Pseudorandom clique size
- (D. Grynkiewicz et al, 2007) Connectivity
- (N. Alon, 2007+) Independence Number
- (B. Cheyne et al 2003; V. Lev, 2008+) Hamiltonicity

Abelian Group Characters

$\chi : \Gamma \to \mathbb{C}$ is a homomorphism.

eg. $\Gamma = Z_5$

- $\chi_1 : x \mapsto e^{2i\pi\frac{x}{5}}$
- $\chi_2 : x \mapsto e^{2i\pi\frac{2x}{5}}$
- $\chi_0 : x \mapsto 1$



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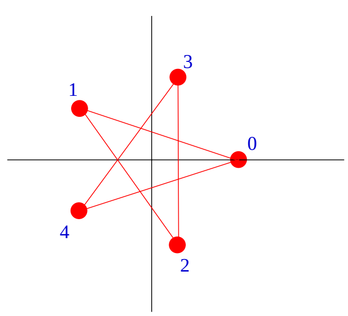

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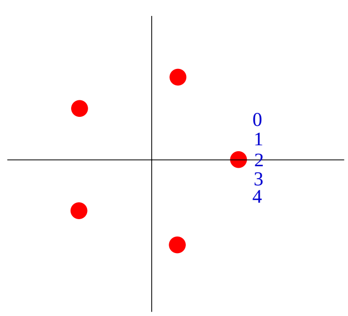

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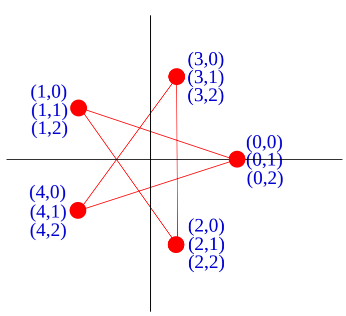
For $a \in Z_n$, $\chi_a(x) = e^{2i\pi \frac{ax}{n}}$

Abelian Group Characters

eg. $\Gamma = Z_5 \times Z_3$

- $\chi_{(2,0)} : (x_1, x_2) \mapsto e^{2i\pi \frac{2x_1}{5}}$
- $\chi_{(0,1)} : (x_1, x_2) \mapsto e^{2i\pi \frac{x_2}{3}}$
- $\chi_{(3,2)} : (x_1, x_2) \mapsto e^{2i\pi (\frac{3x_1}{5} + \frac{2x_2}{3})}$



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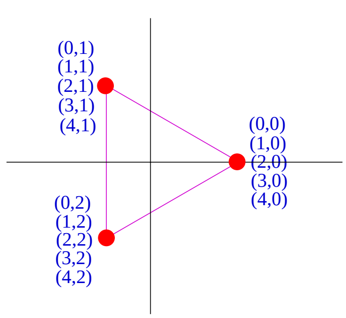

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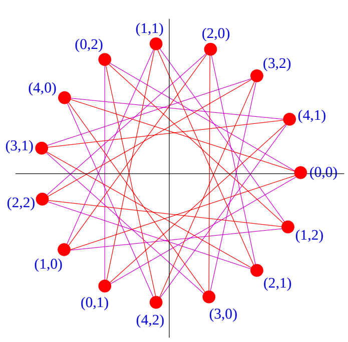

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- $\chi_{(3,2)} : (x_1, x_2) \mapsto e^{2i\pi(\frac{3x_1}{5}+\frac{2x_2}{3})}$

For $(a_1, a_2) \in Z_5 \times Z_3$, $\chi_{(a_1,a_2)}(x_1, x_2) = e^{2i\pi(\frac{a_1x_1}{5}+\frac{a_2x_2}{3})}$

Abelian Group Characters

eg. $\Gamma = Z_5 \times Z_3$

- $\chi_{(2,0)} : (x_1, x_2) \mapsto e^{2i\pi \frac{2x_1}{5}}$
- $\chi_{(0,1)} : (x_1, x_2) \mapsto e^{2i\pi \frac{x_2}{3}}$
- $\chi_{(3,2)} : (x_1, x_2) \mapsto e^{2i\pi (\frac{3x_1}{5} + \frac{2x_2}{3})}$

For $a \in Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$, $\quad \chi_a(x) = e^{2i\pi(\frac{a_1 x_1}{n_1} + \cdots + \frac{a_2 x_2}{n_k})}$

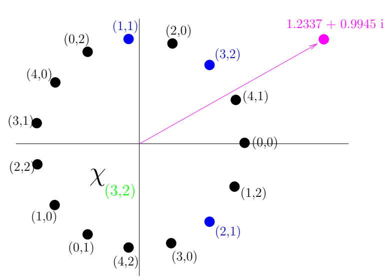
Spectra of Cayley Graphs

Theorem (Classic)

$$Spec(\mathrm{Cay}(\Gamma, \{s_1, \ldots, s_k\})) = \left\{ \sum_i \chi_a(s_i) \;\middle|\; a \in \Gamma \right\}$$

$Cay(Z_5 \times Z_3, \{(1,1), (2,1), (3,2)\})$

$\lambda_{(3,2)} = 1.2337 + 0.9945i$



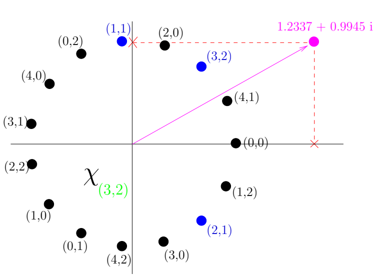
Spectra of Cayley Graphs

Theorem (Classic)

$$Spec(\mathrm{Cay}(\Gamma, \{s_1, \ldots, s_k\})) = \left\{ \sum_i \chi_a(s_i) \mid a \in \Gamma \right\}$$

$Cay(Z_5 \times Z_3, \{(1,1), (2,1), (3,2)\})$

$$\lambda_{(3,2)} = 1.2337 + 0.9945i$$
$$\Re(\lambda_{(3,2)}) = 1.2337$$
$$\Im(\lambda_{(3,2)}) = 0.9945$$



2-to-2 (unless χ_a is real-valued)

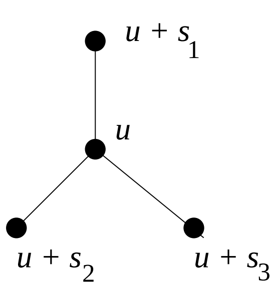
Spectra of Cayley Graphs

Theorem (Classic)

$$Spec(\mathrm{Cay}(\Gamma, \{s_1, \ldots, s_k\})) = \left\{ \sum_i \chi_a(s_i) \;\middle|\; a \in \Gamma \right\}$$

Proof:

$$(A\,\chi_a)(u) = \sum_{v \in N(u)} \chi_a(v)$$
$$= \sum_i \chi_a(u + s_i)$$
$$= \chi_a(u) \sum_i \chi_a(s_i)$$



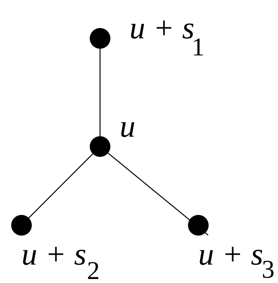
Spectra of Cayley Graphs

Theorem (Classic)

$$Spec(\mathrm{Cay}(\Gamma, \overbrace{\{s_1, \ldots, s_k\}}^{S})) = \left\{ \overbrace{\sum_i \chi_a(s_i)}^{\chi_a(S)} \;\middle|\; a \in \Gamma \right\}$$

Proof:

$$(A\,\chi_a)(u) = \sum_{v \in N(u)} \chi_a(v) = \sum_i \chi_a(u + s_i) = \chi_a(u) \underbrace{\sum_i \chi_a(s_i)}_{\chi_a(S)}$$



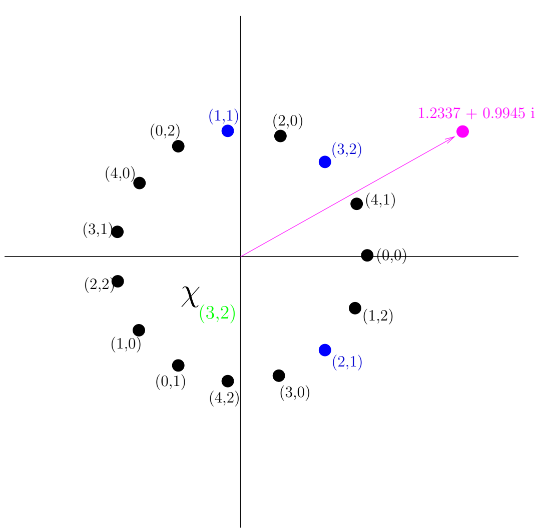
Spectra of Cayley Sum graphs

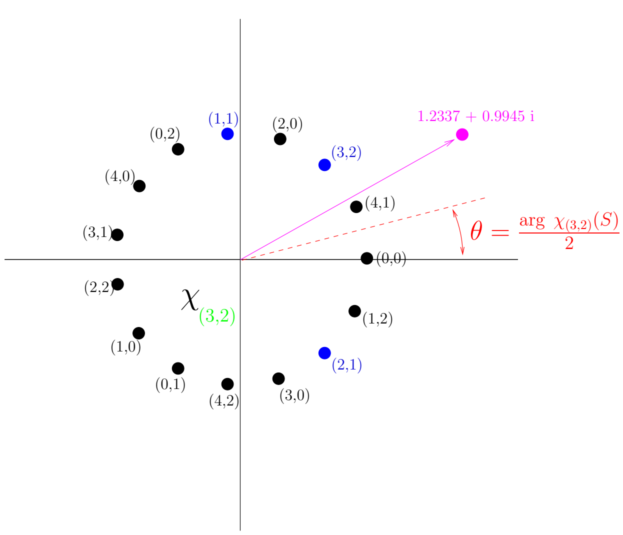
χ_a is real-valued (± 1 - valued) iff $a \in \mathrm{Inv}(\Gamma)$.

Theorem (Chung 89, DGMS)

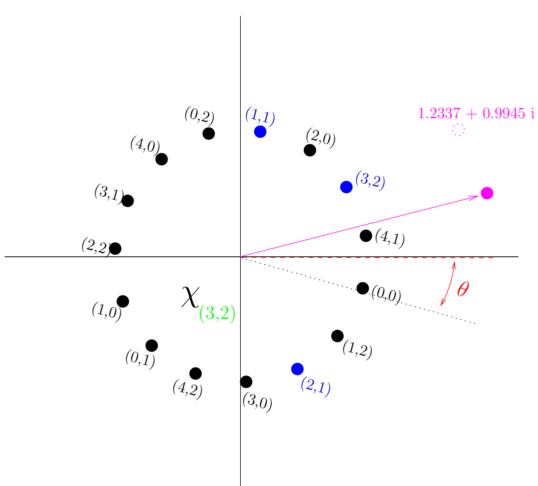
$$
\begin{aligned}
Spec(\mathrm{CaySum}(\Gamma, S)) = \quad & \{ \quad \chi_a(S) \quad | \quad a \in \mathrm{Inv}(\Gamma) \ \} \\
& \cup \{ \pm |\chi_a(S)| \ | \ a, -a \notin \mathrm{Inv}(\Gamma) \ \}
\end{aligned}
$$

Eigenvectors are phase-shifted characters $\chi_a \cdot e^{-\arg(\chi_a(S))/2}$.

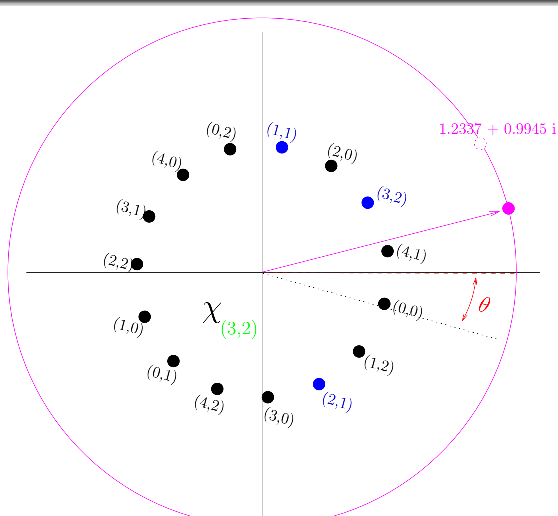

1.2337 + 0.9945 i
(1,1)
(2,0)
(0,2)
(3,2)
(4,0)
(4,1)
(3,1)
(0,0)
(2,2)
$\chi_{(3,2)}$
(1,2)
(1,0)
(2,1)
(0,1)
(4,2)
(3,0)



1.2337 + 0.9945 i
(1,1)
(2,0)
(0,2)
(3,2)
(4,0)
(4,1)
(3,1)
(0,0)
(2,2)
$\chi_{(3,2)}$
(1,2)
(1,0)
(2,1)
(0,1)
(4,2)
(3,0)
$\theta = \frac{\arg\ \chi_{(3,2)}(S)}{2}$



1.2337 + 0.9945 i
(0,2)
(1,1)
(2,0)
(4,0)
(3,2)
(3,1)
(4,1)
(2,2)
θ
(0,0)
(1,0)
χ(3,2)
(1,2)
(0,1)
(4,2)
(3,0)
(2,1)



Eigenvalues: $\pm|1.2337 + 0.9945i| = \pm 2.4423$
2-to-2

Proof:

$$(A\ \chi_a)(u) = \sum_{v \in N(u)} \chi_a(v) = \sum_{s \in S} \chi_a(s-u)$$
$$= \sum_{s \in S} \chi_a(s)\ \chi_a(u)^{-1} = \chi_a(S)\ \overline{\chi_a(u)} \quad \text{Not quite!}$$

Proof: Phase change: $\rho := e^{-\frac{\arg \chi_a(S)}{2}}$. So $\chi_a(S)\rho^2 = |\chi_a(S)|$.

$$(A\,\chi_a)(u) = \sum_{v \in N(u)} \chi_a(v) = \sum_{s \in S} \chi_a(s-u)$$

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$$(A\,\rho\chi_a)(u) = \sum_{v \in N(u)} \rho\chi_a(v) = \sum_{s \in S} \rho\chi_a(s-u)$$
$$= \sum_{s \in S} \rho^2 \chi_a(s)\,(\rho\chi_a(u))^{-1} = |\chi_a(S)|\,\overline{\rho\chi_a(u)}$$

So

$$\Re(\rho\chi_a) \text{ has eigenvalue } +|\chi_a(S)|$$
$$\Im(\rho\chi_a) \text{ has eigenvalue } -|\chi_a(S)|$$

2 to 2 (unless a is an involution).

Proof: Phase change: $\rho := e^{-\frac{\arg \chi_a(S)}{2}}$. So $\rho^2 \chi_a(S) = |\chi_a(S)|$.

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$$\Im(\rho\chi_a) \text{ has eigenvalue } -|\chi_a(S)|$$

2 to 2 (unless a is an involution).

(3,6)-Fullerenes are Cayley Sum Graphs

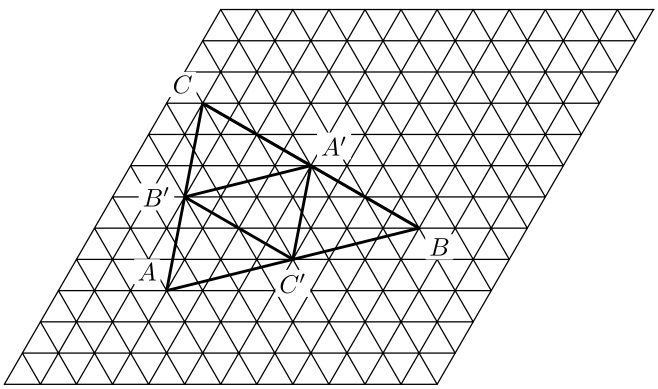
Clincher!

Theorem (DGMS)

Every (3,6)-Fullerene is a Cayley Sum Graph on $\mathbb{Z}_{2m} \times \mathbb{Z}_{2n}$.

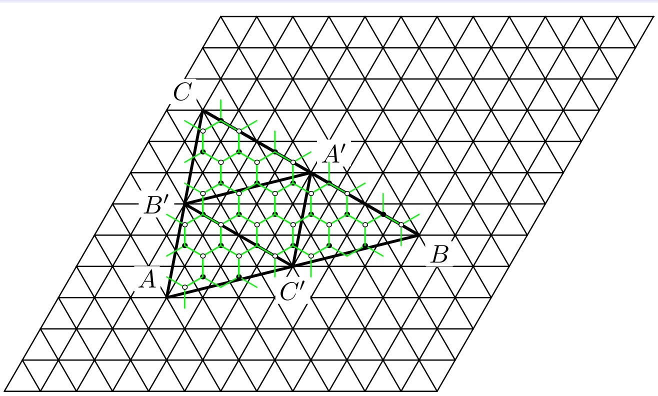
- Involutions $a \in \{(0,0),\ (m,0),\ (0,n),\ (m,n)\}$ yield $\lambda = \chi_a(S) \in \{3, -1, -1, -1\}$
- NonInvolutions $a = \pm(x, y)$ yield eigenvalue pairs $\lambda = \pm|\chi_a(S)|$

Grid Construction of $(3,6)$-Fullerenes



- *ABC* any acute triangle on Eisenstein grid
- Midpoints A', B', C' are grid points
- Fold tetrahedron
- Take the dual graph

Grid Construction of $(3,6)$-Fullerenes



- *ABC* any acute triangle on Eisenstein grid
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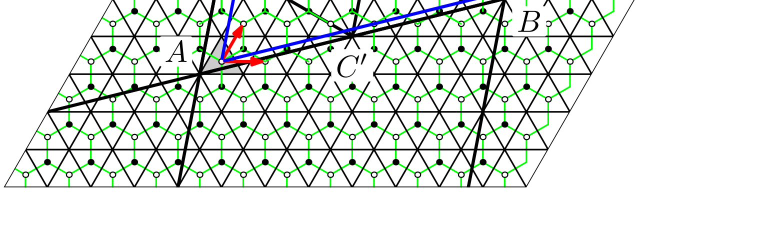
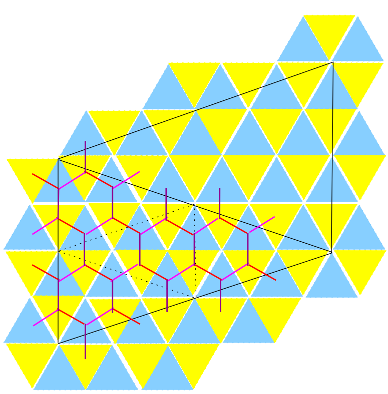
Proof Setup

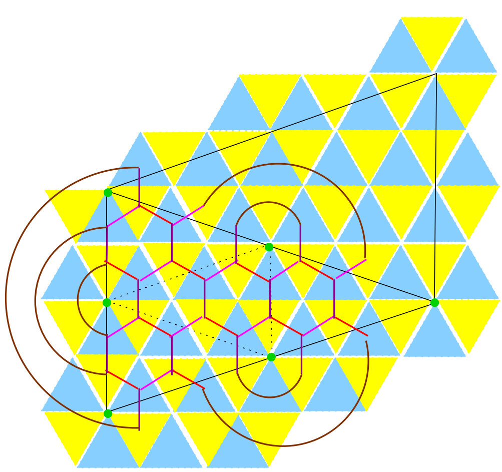


Proof Setup



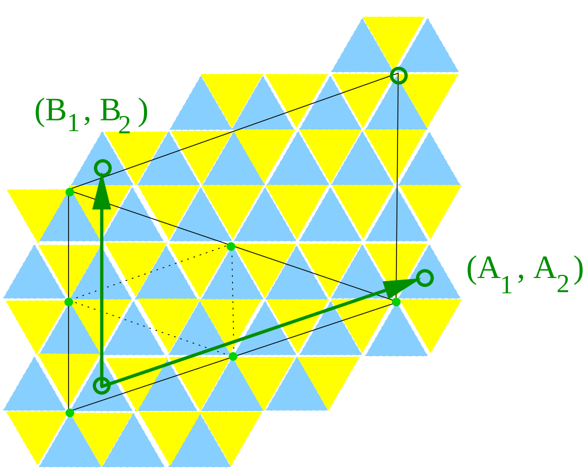
Vertices are Up-triangles and Down-triangles

Proof Setup



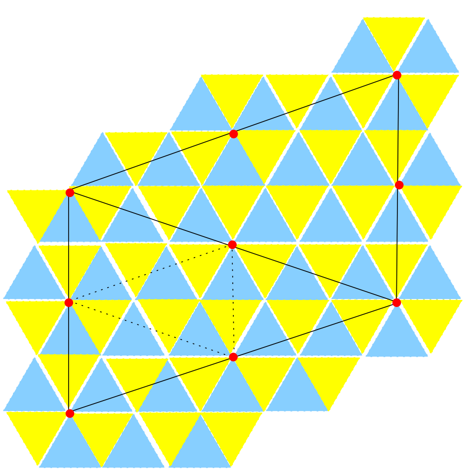
Folding identifies vertices and connects edges

Proof Setup



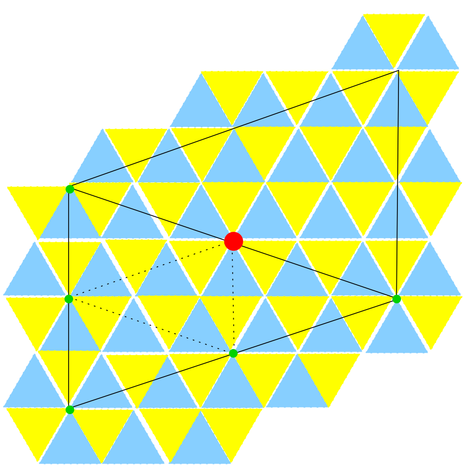
$A = (A_1, A_2)$ and $B = (B_1, B_2)$ generate lattice $\mathcal{L}$

Proof Setup



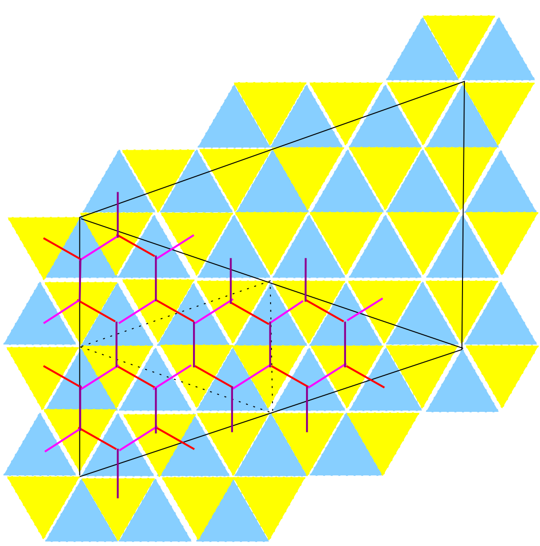
Translated lattice of pivot points

Proof Setup

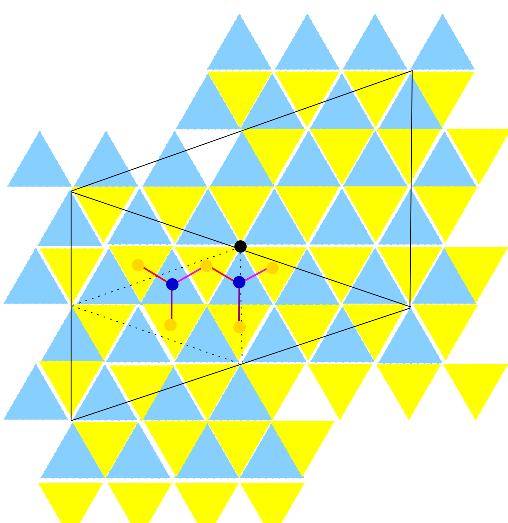


Reflect (rotate) down-triangles about any pivot point

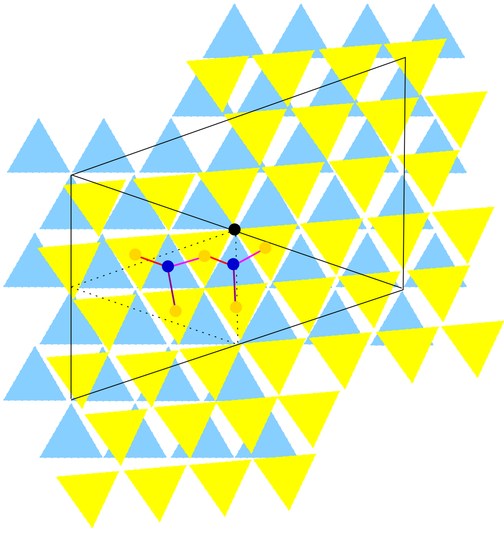
Universal Cover is an Eisenstein Cayley Sum Graph



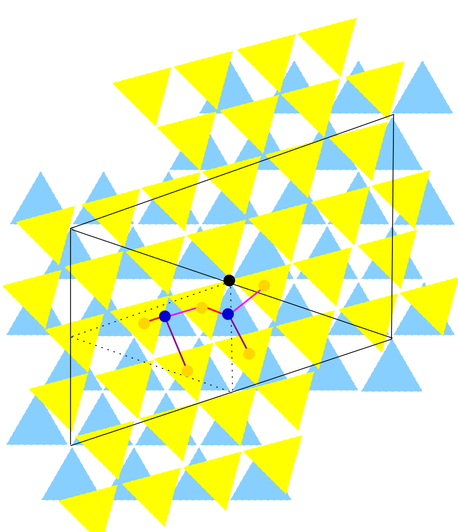
Universal Cover is an Eisenstein Cayley Sum Graph



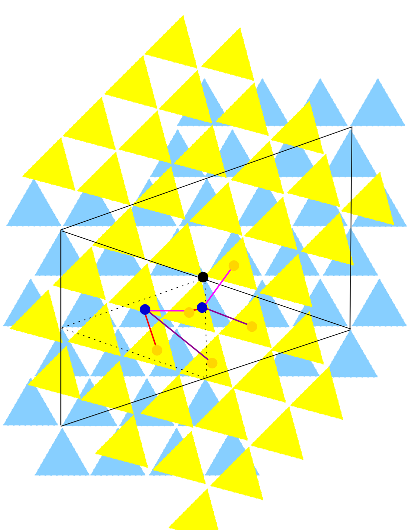
Universal Cover is an Eisenstein Cayley Sum Graph



Universal Cover is an Eisenstein Cayley Sum Graph



Universal Cover is an Eisenstein Cayley Sum Graph



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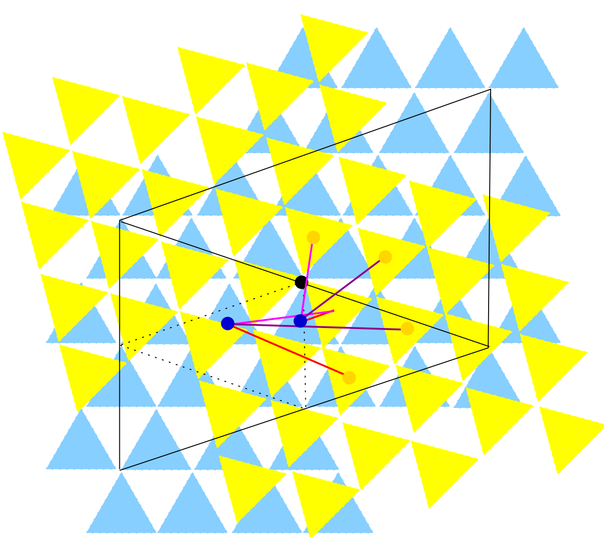

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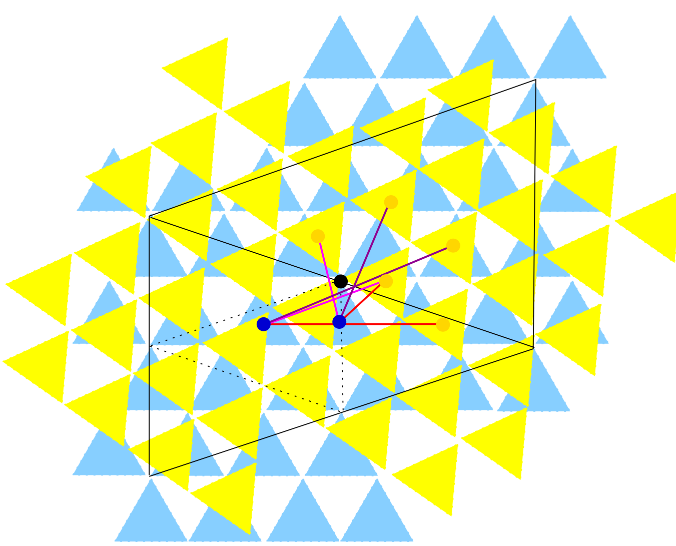
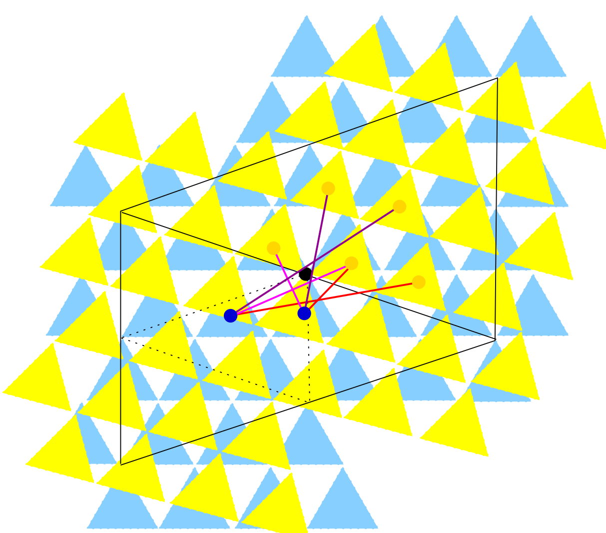
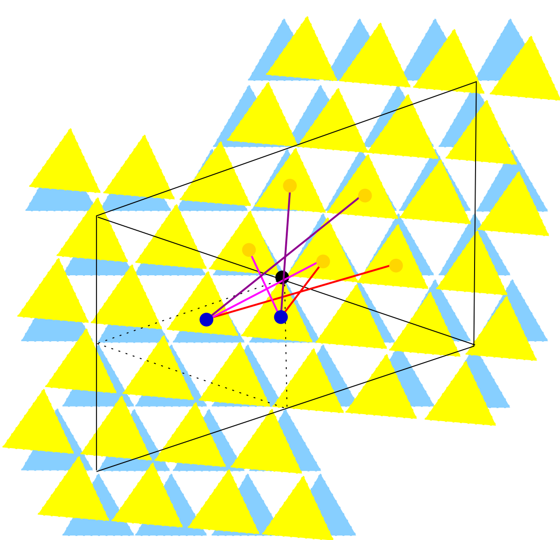


Universal Cover is an Eisenstein Cayley Sum Graph



Universal Cover is an Eisenstein Cayley Sum Graph



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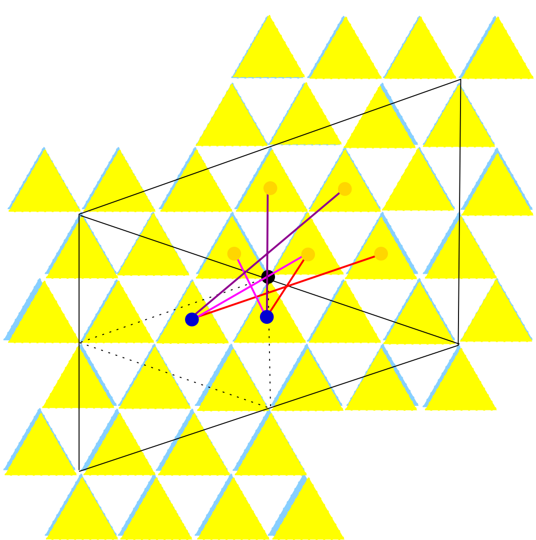
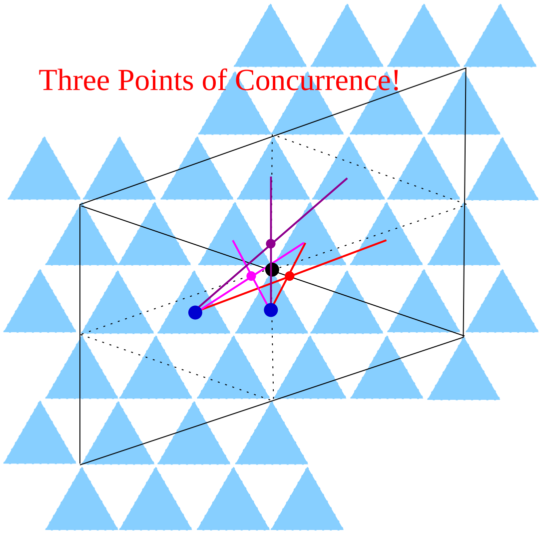
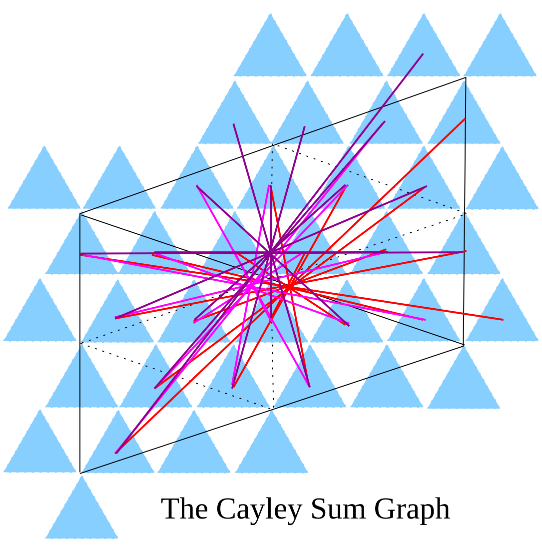


Universal Cover is an Eisenstein Cayley Sum Graph

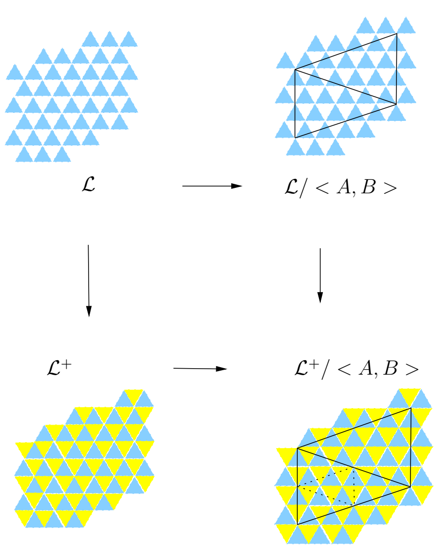


Universal Cover is an Eisenstein Cayley Sum Graph

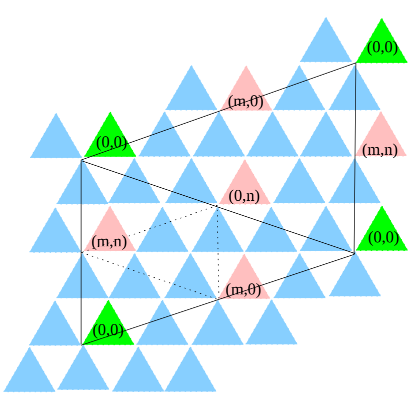


The Cayley Sum Graph

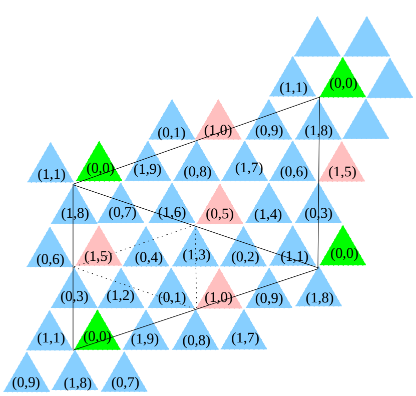
General Procedure



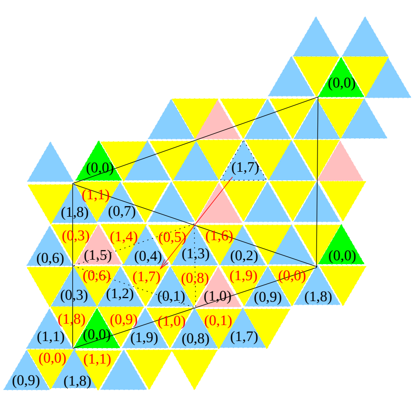
Finding the Homomorphism $\mathbb{E} \to \mathcal{L} \to \mathbb{Z}_{2m} \times \mathbb{Z}_{2n}$



Finding the Homomorphism $\mathbb{E} \to \mathcal{L} \to \mathbb{Z}_{2m} \times \mathbb{Z}_{2n}$

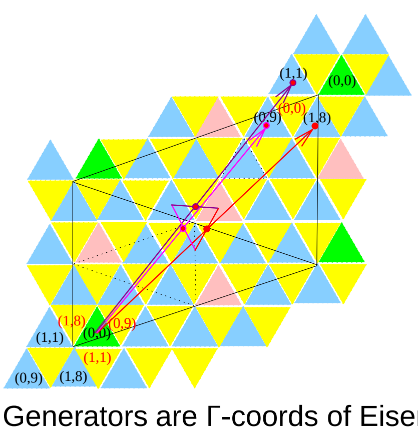


Finding the Homomorphism $\mathbb{E} \longrightarrow \mathcal{L} \longrightarrow \mathbb{Z}_{2m} \times \mathbb{Z}_{2n}$



Rotate

Finding the Homomorphism $\mathbb{E} \longrightarrow \mathcal{L} \longrightarrow \mathbb{Z}_{2m} \times \mathbb{Z}_{2n}$



Generators are Γ-coords of Eisenstein $[-1, 0], [0, -1], [-1, -1]$

Computing It: Lattice Basis Change

- $\mathbb{E} = [a, b] \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix}$ $\quad AB = 4a + 2b \quad AC = -2a + 4b$
- Put $M = \begin{pmatrix} 4 & -2 \\ 2 & 4 \end{pmatrix}$
- *SNF*: $UMV = D$, (for some U, V unimodular, D diagonal)

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 & -2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}$$

- Hence $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_{10}$.
- If $U = (u_1\ u_2)$, then

$$S = \{-u_1, -u_2, -u_1 - u_2\} = \{(1, 8), (1, 1), (0, 9)\}$$

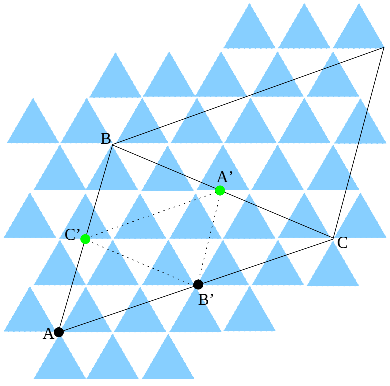
Computing an eigenvalue

Let $a = (1,3) \notin \text{Inv}(\mathbb{Z}_2 \times \mathbb{Z}_{10})$, say.
$S = \{(1,8),(1,1),(0,9)\}$

$$\begin{aligned}\chi_{(1,3)}(S) &= e^{\frac{2i\pi 1(1)}{2}+\frac{2i\pi 3(8)}{10}} + e^{\frac{2i\pi 1(1)}{2}+\frac{2i\pi 3(1)}{10}} + e^{\frac{2i\pi 1(0)}{2}+\frac{2i\pi 3(9)}{10}} \\ &= (2\cos(\pi/5) - \cos 2\pi/5) - i \sin 2\pi/5 \\ &= 1.618033989\, e^{-\pi/5}\end{aligned}$$

Corresponding eigenvalues: $\pm|\chi_{(1,3)}(S)| = \pm 1.618033989$
Phase shift for eigenvectors: $\frac{-1}{2}\arg(\chi_{(1,3)}(S)) = -\pi/10$.

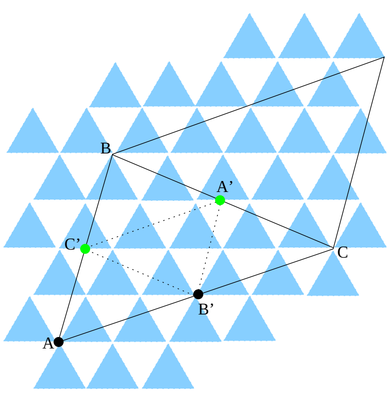
What if A', B', C' are not gridpoints?

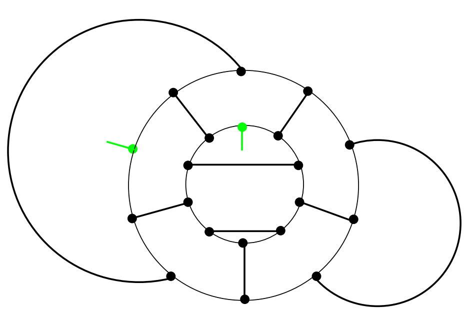


Semiloops!

$(0, 3, 6)$-Fullerene: 3-regular, planar, hexagons & triangles & (up to 4) semiloops

What if A', B', C' are not gridpoints?



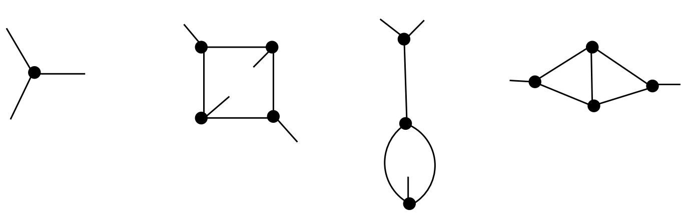


(5, 2, 1)

Semiloops!

(0, 3, 6)-Fullerene: 3-regular, planar, hexagons & triangles & (up to 4) semiloops

Some $(0, 3, 6)$-Fullerenes



$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Eigenvalues = $\{3, 1, -1, -1\}$.

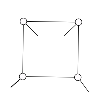
Fowler's other conjecture

Conjecture (Fowler et al., 2002)

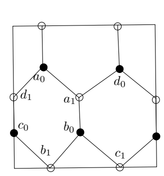
The spectrum of every $(0, 3, 6)$*-Fullerene has the form* $A \cup \Lambda \cup -\Lambda$*, where A contains one 3 and at most three* ± 1*.*

This is true! They are all Cayley Sum Graphs.

Proof for $(0, 3, 6)$-polyhedra

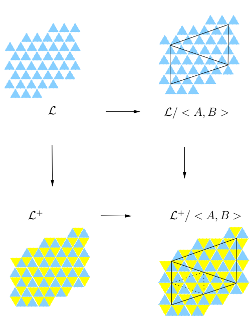


- G — a $(0, 3, 6)$-polyhedron
- $G \times K_2$ — bipartite double cover
- embedding of G gives an embedding of $G \times K_2$ (on a different surface)
- this embedding has all faces of size 6,



  hence it is on torus
- next we proceed as for $(3, 6)$-polyhedra:
  - $\mathcal{U}$—black vertices of the universal cover (infinite hex. grid)
  - $\mathcal{L}$—black vertices that correspond to a given vertex of G
  - etc.

Other Possibilities



- $\mathcal{L} = A_2 \quad \mathcal{L}^+ = A_2^+$: triangular grid ((3,6)-Fullerenes)
- $\mathcal{L} = D_2 \quad \mathcal{L}^+ = Z^2$: checkerboard (requires 4 semiloops)
- $\mathcal{L} = D_3 \quad \mathcal{L}^+ = D_3^+$: diamond packing (8 unpaired eigenvalues)

Summary

- We proved Fowler's conjecture and described precisely eigenvalues and eigenvectors of $(0, 3, 6)$-polyhedra.
- Other lattices yield families with $A \cup \Lambda \cup -\Lambda$
- Other natural families of spectrally nearly bipartite graphs?
- Much Harder: $(5, 6)$-Fullerenes. One problem: When are there more negative than positive eigenvalues?