Convergence Problems for Trigonometric Series

Dr. Songping Zhou

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Dr. S. P. Zhou

Zhejiang Sci-Tech University

James Chair, St Francis Xavier University

HISTORY

- Pure Mathematics consists of Algebra, Geometry and Analysis

- Analysis belongs to Modern Mathematics and relates closely to Motion

HISTORY

- Mathematics: Four key factors:

  Definition, Operation, Computation, Application

- In calculus, any others can be regarded as applications of Limit Theory

HISTORY

- No computation for series?
- Then consider convergence instead!

HISTORY

- The most important series are series which provide bases for a space

Taylor series, Fourier series, wavelets ……

HISTORY

- Fourier series have many important applications in Science and Technology

CONVERGENCE OF FOURIER SERIES

- Pointwise Convergence
- Almost Everywhere Convergence
- Uniform Convergence
- Mean Convergence

Pointwise Convergence

Dini's Criteria. Let $f \in L_{2\pi}$, $g_{x,c}(t) = f(x+t) + f(x-t) - 2c$. If $g_{x,c}(t)/t \in L_{2\pi}$, then $\lim\limits_{n\to\infty} S_n(f,x) = c$.

Jordan's Criteria. Let $f \in L_{2\pi}$ is a bounded variation function on the interval $[a,b]$, then its Fourier series converges to

$$\frac{1}{2}(f(x+0) + f(x-0))$$

at every point in (a,b).

Almost Everywhere Convergence

Fejér's Theorem. There is a 2π periodic and continuous function such that its Fourier series does not converge at some points.

Lusin's Conjecture 1913. If $f \in L^2_{2\pi}$, Then its Fourier series converges almost everywhere.

Carleson (1966)-Hunt(1968) Theorem. If $f \in L^p_{2\pi}$ for $p > 1$, Then its Fourier series converges almost everywhere.

Uniform Convergence

Chaundy-Jolliffe's Theorem 1916. Suppose that $\{a_n\}$ is a non-negative and non-increasing real sequence with $\lim_{n\to\infty} a_n = 0$. Then a necessary and sufficient condition for the uniform convergence of series

$$\sum_{n=1}^{\infty} a_n \sin nx$$

is

$$\lim_{n\to\infty} n a_n = 0.$$

Mean Convergence

L^1-Convergence.W. H. Young 1913, Kolmogorov 1923. Let $f \in L_{2\pi}$ be an even function and $\sum_{n=1}^{\infty} a_n \cos nx$ be its Fourier series. Suppose that $\{a_n\}$ is a nonnegative and non-increasing real sequence. Then a necessary and sufficient condition for

$$\lim_{n\to\infty} \|f - S_n(f)\|_L = 0$$

is

$$\lim_{n\to\infty} a_n \log n = 0.$$

Classes of Sequences

MS: The class of nonnegative, non-increasing sequences.

CQMS: a sequence $\{a_n\}$ is a quasimonotone sequence (in symbol, $\{a_n\} \in \mathbf{CQMS}$) if there is an $\alpha \geq 0$ such that a_n/n^α is decreasing for all n.

(S. M. Shah 1947, O. Szasz 1948)

RVQMS: Let $R(n)$ be an increasing sequence with $R(2n)/R(n)$ bounded. A sequence $\{a_n\}$ is said to be O-regularly varying quasi-monotone sequence ($\{a_n\} \in \mathbf{RVQMS}$) if for some $R(n)$ with the above properties, $a_n/R(n)$ is decreasing for all n.

(O-regularly varying sequence: J. Karamata 1930)

Classes of Sequences

RBVS: A nonnegative sequence $\mathbf{A} = \{a_n\}$ with $\lim_{n\to\infty} a_n = 0$ is said to be a "rest bounded variation" sequence ($\{a_n\} \in \mathbf{RBVS}$) if

$$\sum_{k=n}^{\infty} |a_k - a_{k+1}| \leq M(\mathbf{A})a_n$$

holds for some constant $M(\mathbf{A})$ depending only upon the sequence $\mathbf{A}$ and all $n = 1, 2, \cdots$.

(L. Leindler 2001)

AMS: A nonnegative sequence $\{b_n\}$ is said to be "*almost monotone*" (in symbol, $\{b_n\} \in \mathbf{AMS}$) if there is a positive constant $M(\mathbf{b})$ such that

$$b_k \leq M(\mathbf{b})b_n \quad \text{for all} \quad k \geq n.$$

(S. N. Bernstein about 1912)

Classes of Sequences

GBVS: A nonnegative sequence $\mathbf{A} = \{a_n\}$ is said to be "group bounded variation" sequence ($\{a_n\} \in \mathbf{GBVS}$) if

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \leq M(\mathbf{A}) a_n$$

holds for some constant $M(\mathbf{A})$ and all $n = 1, 2, \cdots$.

(R. J. Le and S. P. Zhou 2005)

Classes of Sequences

NBVS: A nonnegative sequence $\mathbf{A} = \{a_n\}$ is said to be "non-onesided bounded variation sequence" ($\{a_n\} \in \mathbf{NBVS}$) if

$$\sum_{k=n}^{2n} |\Delta a_n| \le M(\mathbf{A})(a_n + a_{2n})$$

holds for all $n = 1, 2, \cdots$.

(D. S. Yu and S. P. Zhou 2007)

Classes of Sequences

What is the Ultimate Generalization to Monotonicity?

MVBVS (Mean Value Bounded Va

AMS is not A Right Class

Theorem 1. There exists a sequence $\{b_n\} \in$ AMS with $\lim_{n\to\infty} nb_n = 0$ such that the series $\sum_{n=1}^{\infty} b_n \sin nx$ is not uniformly convergent.

MVBVS Essentially Generalizes all Known Classes

Proposition 1. If $\mathbf{A} = \{a_n\} \in$ GBVS in general sense, i.e., $\{a_n\}$ satisfies

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \leq C(\mathbf{A}) \max_{n \leq k < n+N_0} a_k \qquad (1)$$

for some given $N_0 \geq 1$, then $\{a_n\} \in$ MVBVS. But the reverse is not true, i.e. there are sequences in MVBVS not satisfying (1).

Proposition 2. If $\mathbf{A} = \{a_n\} \in$ NBVS, then $\{a_n\} \in$ MVBVS. But the reverse is not true, i.e. there are sequences in MVBVS which are not in NBVS.

Uniform Convergence

Theorem 2. If $\mathbf{A} = \{a_n\} \in$ MVBVS, then a necessary and sufficient condition either for the uniform convergence of series $\sum_{n=1}^{\infty} a_n \sin nx$, or for the continuity of its sum function f, is that $\lim_{n\to\infty} na_n = 0$.

L^1-Convergence

Theorem 3. Let $f(x) \in L_{2\pi}$ be a complex valued function. If the Fourier coefficients $\hat{f}(n)$ of f satisfy that $\{\hat{f}(n)\}_{n=0}^{+\infty} \in MVBVS$ and

$$\lim_{\mu \to 1+0} \limsup_{n \to \infty} \sum_{k=n}^{[\mu n]} |\Delta \hat{f}(k) - \Delta \hat{f}(-k)| \log k = 0.$$

Then

$$\lim_{n \to \infty} \|f - S_n(f)\|_L = 0$$

if and only if

$$\lim_{n \to \infty} \hat{f}(n) \log |n| = 0.$$

Class MVBVS

MVBVS: A nonnegative sequence $\mathbf{A}=\{a_n\}_{n=0}^{\infty}$ is said to be a *mean value bounded variation sequence* ($\{a_n\}\in$ MVBVS) if there is a $\lambda\geq 2$ such that

$$\sum_{k=n}^{2n}|a_k-a_{k+1}|\leq\frac{C(\mathbf{A})}{n}\sum_{k=[\lambda^{-1}n]}^{[\lambda n]}a_k$$

holds for all $n=1,2,\cdots$ and some constant $C(\mathbf{A})$ depending only upon the sequence $\mathbf{A}$.

(S. P. Zhou, P. Zhou and D. S. Yu 200?)

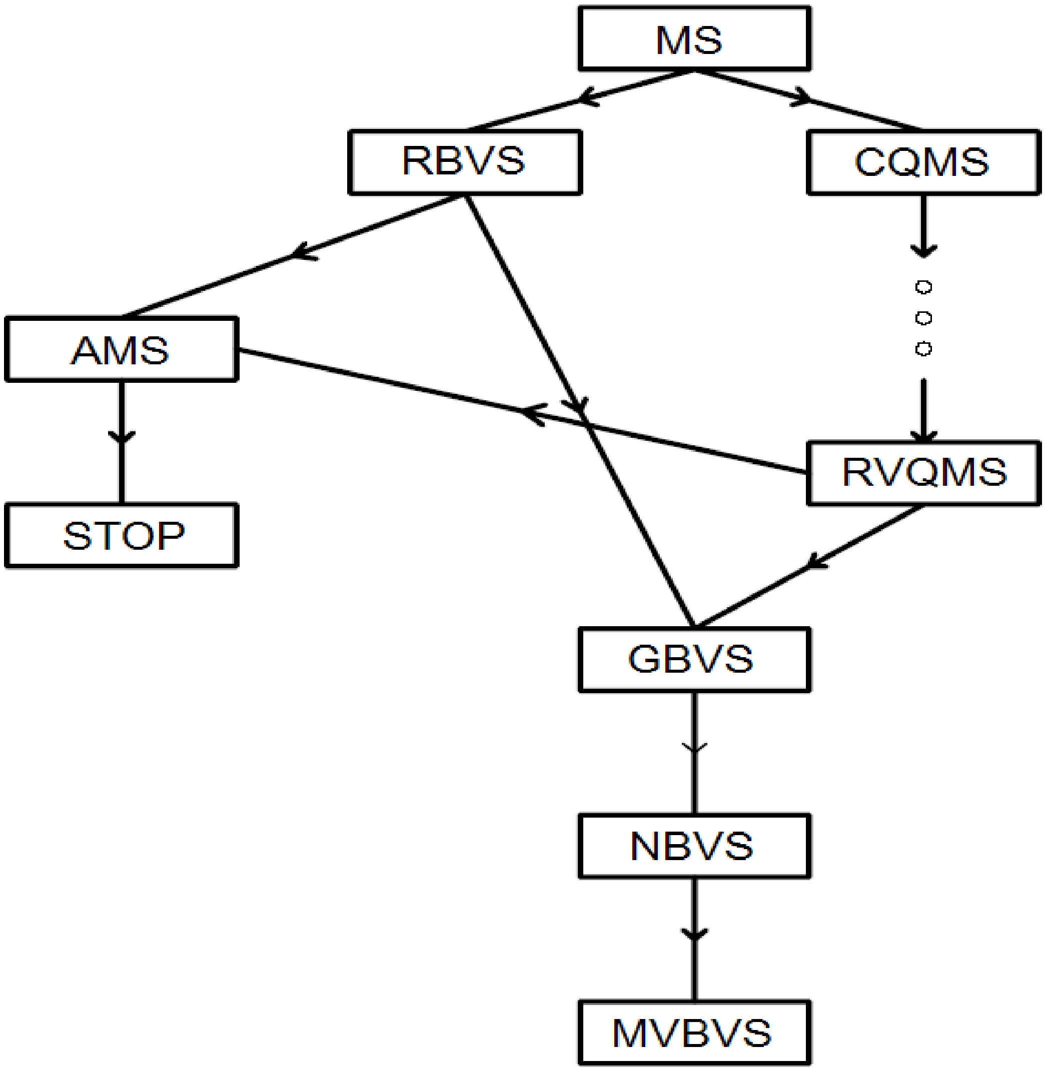
The Ultimate Generalization to Monitonicity

Theorem 4. Let $\{M_n\}$ be a given nonnegative increasing sequence tending to infinity. Then there exists a sine series of the form $\sum_{n=1}^{\infty} a_n \sin nx$ with $\lim_{n\to\infty} na_n = 0$ such that for any given $\lambda \geq 2$,

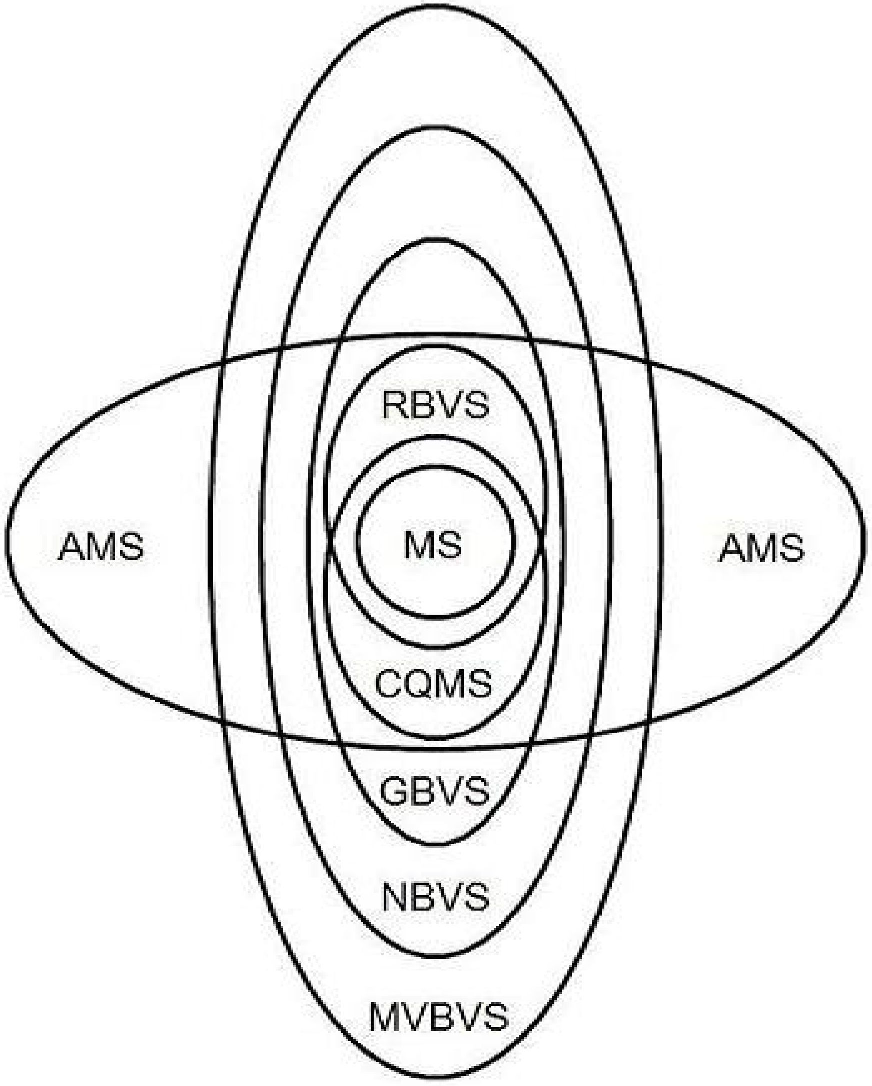
$$\lim_{n\to\infty} \frac{\sum_{k=n}^{2n} |\Delta a_k|}{\frac{M_n}{n} \sum_{k=[\lambda^{-1}n]}^{[\lambda n]} a_k} = 0,$$

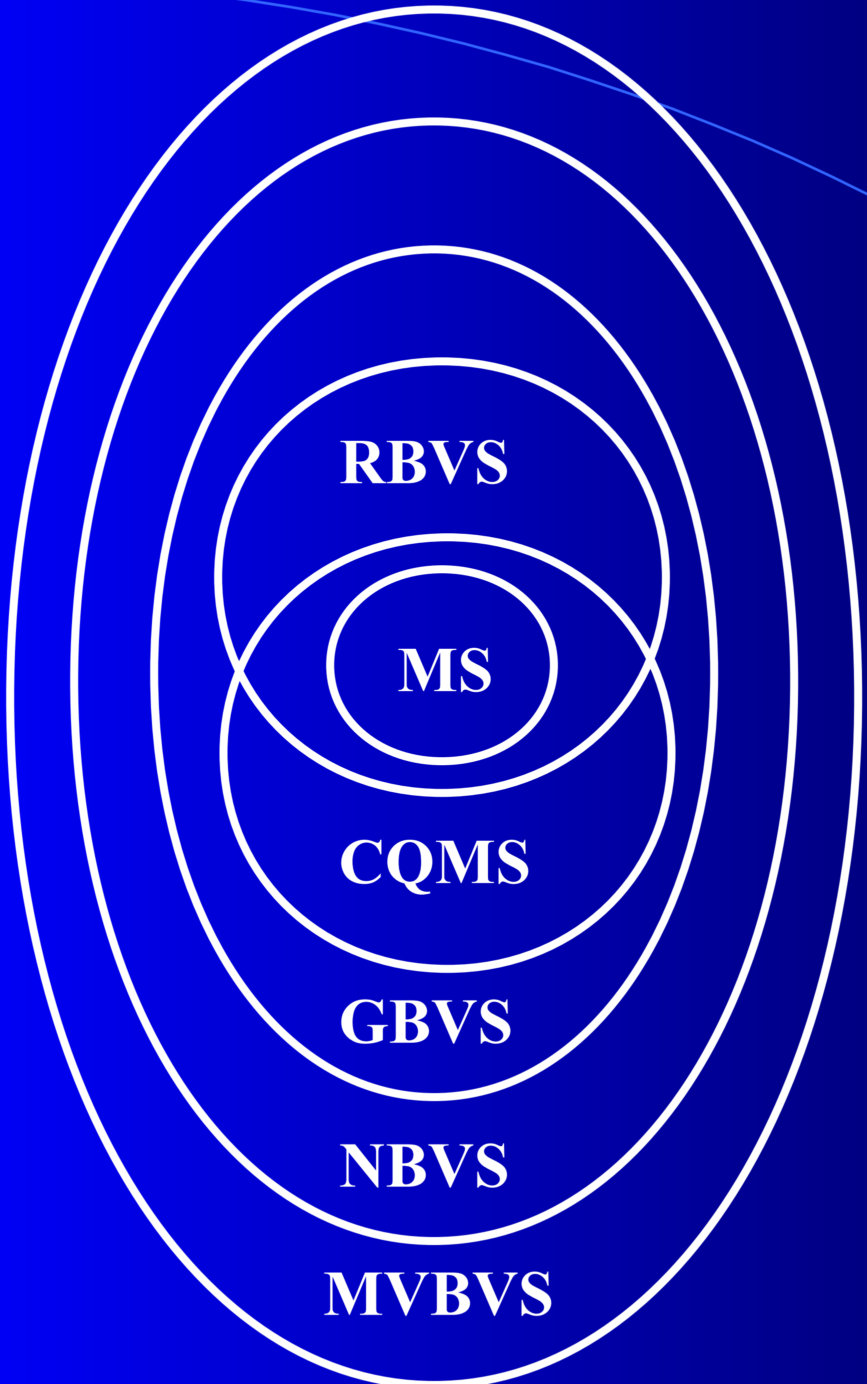
however, the series is not uniformly convergent.

Development History



Relationship of Classes





- **MS:** 单调递减数列

(1916,1913,1923)

- **CQMS:** 拟单调数列

(1947,1948)

- **RBVS:** 剩余有界变差数列

(2001)

- **GBVS:** 分组有界变差数列

(2005)

- **NBVS:** 非单边有界变差数列

(2007)

- **MVBVS:** 均值有界变差

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Ultimate Generalization to Monotonicity for Uniform Convergence of Trigonometric Series

Song Ping Zhou ,[1]
Institute of Mathematics, Zhejiang Sci-Tech University, Xiasha Economic Development Area, Hangzhou, Zhejiang 310018 China, and
Department of Mathematics, Statistics & Computer Science, St. Francis Xavier University, Antigonish, Nova Scotia, Canada B2G 2W5
szhou@zjip.com
Ping Zhou,[2] Dan Sheng Yu [3]
Department of Mathematics, Statistics & Computer Science, St. Francis Xavier University, Antigonish, Nova Scotia, Canada B2G 2W5
pzhou@stfx.ca, dyu@stfx.ca

Abstract

Chaundy and Jolliffe [4] proved that if $\{a_n\}$ is a non-increasing (monotonic) real sequence with $\lim\limits_{n\to\infty} a_n = 0$, then a necessary and sufficient condition for the uniform convergence of the series $\sum_{n=1}^{\infty} a_n \sin nx$ is $\lim\limits_{n\to\infty} na_n = 0$. We generalize (or weaken) the monotonic condition on the coefficient sequence $\{a_n\}$ in this classical result to the so-called mean value bounded variation condition and prove that the generalized condition cannot be weakened further. We also establish an analogue to the generalized Chaundy and Jolliffe theorem in the complex space.

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- D. S. Yu, P. Zhou and S. P. Zhou: On L^1 convergence of Fourier series under MVBV condition, Canad. Math. Bull. , to appear.
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Thank You