Paley's Theorem for Hankel Matrices via the Schur Test

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Abstract. Paley's theorem for lacunary coefficients of H^1-functions is equivalent to a statement about lacunary Hankel matrices acting on ℓ^2 of the non-negative integers. That statement reduces easily to the case where the entries in the matrix are all non-negative. So it must be provable by the Schur test. We give such a proof, with an interesting pattern in the vector used in the test.

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Paley-Hankel. Consider matrices $(A(m,n))_{m,n=0}^{\infty}$ that have the *Hankel symmetry* where the entries only depend on the sum of the indices.

Examples.

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

$$A_v = \begin{bmatrix} v(0) & v(1) & 0 & v(2) & \cdots \\ v(1) & 0 & v(2) & 0 & \cdots \\ 0 & v(2) & 0 & 0 & \cdots \\ v(2) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & v(3) & \cdots \\ 0 & 0 & v(3) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem PH. *If $v \in \ell^2$, and the gaps in the Hankel matrix A_v grow geometrically, then that matrix acts boundedly on ℓ^2.*

Mixed norms. In Theorem PH, only the size of the numbers $v(m)$ matters. So proving the theorem reduces to the case where $v \geq 0$.

It should be easier to decide whether a matrix acts boundedly on ℓ^2 when the matrix entries are nonnegative.

In any case, it suffices for the *Schur* (Hadamard) product, $A \star A$ say, with entries $(A(m,n)^2)$ to be majorized by the Schur product of two matrices, B and C say, that satisfy the opposite mixed-norm conditions

$$\sup_m \left\{ \sum_n |B(m,n)| \right\} < \infty,$$

and

$$\sup_n \left\{ \sum_m |C(m,n)| \right\} < \infty.$$

Theorem GK. *When the entries in A are nonnegative, the existence of such a majorization, and more, is also necessary for A to act boundedly on ℓ^2.*

More. In that case, the conditions below must also be satisfied.

Schur Test. *There are vectors, u and u' say, with strictly positive entries, and a constant c, so that*

$$Au \leq cu' \quad \text{and} \quad A^T u' \leq cu. \tag{$\dagger$}$$

Then $A \star A$ is equal to $B \star C$, where

$$B(m,n) = A(m,n)\frac{u(n)}{u'(m)}$$

and

$$C(m,n) = A(m,n)\frac{u'(m)}{u(n)}$$

for all indices m and n.

When the matrix A is symmetric, condition ($\dagger$) reduces to having

$$Aw \leq cw \tag{$\dagger\dagger$}$$

for some strictly positive vector w.

Example. For the matrix H, use

$$w(n) = \frac{1}{(n+1)^\alpha}$$

for any power α in the open interval $(0, 1)$.

Problem JJFF. *Find a vector w that works for the matrix A_v when the gap condition is satisfied, $v \in \ell^2$, and $v \geq 0$.*

Problem GB. *Do the same for a general matrix that acts boundedly on ℓ^2 and has only nonnegative entries.*

Majorizing factors.

The matrix $A_v \star A_v$ is equal to the Schur product of the matrix

$$B_v = \begin{bmatrix} v(0) & v(1)^2 & 0 & v(2)^2 & \cdots \\ 1 & 0 & v(2)^2 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and its transpose

$$C_v = \begin{bmatrix} v(0) & 1 & 0 & 1 & \cdots \\ v(1)^2 & 0 & 1 & 0 & \cdots \\ 0 & v(2)^2 & 0 & 0 & \cdots \\ v(2)^2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & v(3)^2 & \cdots \\ 0 & 0 & v(3)^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Boundedness. Since $v \in \ell^2$, the specified mixed-norm conditions hold if and only there is a uniform bound, K say, on the number of 1's in rows of B_v.

That happens if and only if for each index n the first row of the matrix A_v has most K nonzero entries with indices in the interval $[n, 2n)$.

That sparsity condition is necessary for the conclusion of Theorem PH to hold for all sequences v in ℓ^2.

The vector pattern.

When $k_0 = 0$, and $k_{j+1} = 1 + 2k_j$ for all j, each nonnegative integer n has exactly one representation as an alternating sum

$$k_{j_1} - k_{j_2} + k_{j_3} - \cdots \pm k_{j_P},$$

where $0 < j_1 < j_2 \cdots < j_{2P+1}$ and P is a nonnegative integer.

In that case, let

$$w(n) = \prod_{p=1}^{P} v(j_p), \tag{W}$$

with the usual conventions about empty sums and products.

Then

$$B_v(m,n) = A_v(m,n)\frac{w(m)}{w(n)} \quad \text{for all} \quad (m,n)$$

provided that $v(k) \neq 0$ for all $k > 0$.

A recursion. Let w_0 be the vector $(1, 0, 0, \cdots)^T$, and let $A_v^{(k)}$ be the matrix with all 0 entries except for the value $v(k)$ on the antidiagonal where $m+n = j_k$. For each integer $k > 0$, let

$$w_k = A_v^{(k)} \left(\sum_{i=0}^{k-1} w_i \right).$$

The vectors w_k have disjoint supports, and

$$w = \sum_{k=0}^{\infty} w_k = \prod_{k=1}^{\infty} \left[I + A_v^{(k)} \right] w_0 \qquad (\pi)$$

with the successive factors in the product inserted on the left.

Or, let R_k denote reflection at $(1/2)j_k$. Then

$$w_k = v(k) R_k \left(\sum_{i=0}^{k-1} w_i \right).$$

Except for the factor $v(k)$, this follows the Thue-Morse pattern.

How a Hankel matrix act on a vector. Consider a Hankel matrix A with first row a in $\ell^2(Z_+)$ and a vector c also in $\ell^2(Z_+)$.

Extend c to be 0 on the rest of Z, and then reverse it to get $\check{c}$ say. Extend a in the same way, but do *not* reverse it. Convolving the sequences a and $\check{c}$ and restricting the result to Z_+ gives the vector Ac.

In the case (π) above, the vector w comes from suitably shifting certain initial parts of w, rescaling those shifted copies suitably, folding them back towards 0, and appending that part to the previous initial part.

In general, if A acts on ℓ^2 with norm less than 1, then the series

$$\sum_{k=0}^{\infty} A^k w_0$$

converges in ℓ^2. If the entries in A are all nonnegative, the sum, u say, of that series has nonnegative entries, and $Au \le u$.

Paley. Denote the unit circle in the complex plane by T, and identify it with the interval $(-\pi, \pi]$ under the mapping $t \mapsto e^{it}$ with the normalized measure $dt/2\pi$. Given a function f in $L^1(T)$, denote its Fourier coefficients by $\hat{f}(n) = \int_T f(t)e^{-int}\,(dt/2\pi)$.

The classical space $H^1(U)$, originally defined as a set of analytic functions in the unit disk, can be identified with the corresponding space of boundary-values on T, namely those functions f in $L^1(T)$ with the property that $\hat{f}(n) = 0$ for all $n < 0$.

Theorem P. If nonnegative integers $(k_j)_{j=0}^{\infty}$ satisfy a lacunarity condition of the form $k_{j+1} > 2k_j$ for all values of j, then

$$\sum_{j=0}^{\infty} |\hat{f}(k_j)|^2 \leq \{c\|f\|_1\}^2$$

for all functions f in $H^1(T)$.

That is, restricting the coefficients of H^1 functions to such a lacunary set $K = \{k_j\}_{j=0}^{\infty}$ maps the space $H^1(T)$ into $\ell^2(K)$.

Duality. By Hahn-Banach, this is equivalent to the following statement.

Theorem R. *Let K be a lacunary set of nonnegative integers, and let $v \in \ell^2$. Then there exists a function g in $L^\infty(T)$ with $\hat{g}(k_j) = v_j$ for all j, while $\hat{g}(n) = 0$ for all other nonnegative integers n.*

There is no restriction here on $\hat{g}(n)$ when n is negative. One can require that $\|g\|_\infty \le c(K)\|v\|_2$.

Observation F. *The Schur algorithm yields the coefficients of functions with these properties.*

Paley too. A variant of that construction makes $\hat{g}$ vanish at all negative integers n, rather than vanishing at the other nonnegative integers.

A simpler construction with that property had already been discovered twice. Since $k_{j+1} > 2k_j$ for all j, each integer n has at most one representation as an alternating sum

$$k_{j_1} - k_{j_2} + k_{j_3} - \cdots - k_{j_{2P}} + k_{j_{2P+1}},$$

where $j_1 < j_2 \cdots < j_{2P+1}$ and P is a nonnegative integer. In that case, let

$$w(n) = (-1)^P \prod_{p=1}^{2P+1} v(j_p),$$

and let $w(n) = 0$ otherwise. Compare formula (W).

Theorem CF. *These numbers $w(n)$ are the coefficients of a bounded function g.*

Theorem P2. *If $f \in L^1(T)$ and $\hat{f}(n) = 0$ for all positive integers n that are not in the lacunary set K, then $\sum_{j=0}^{\infty} |\hat{f}(k_j)|^2 \leq \{\tilde{c}(K)\|f\|_1\}^2$.*

Symbol. If a Hankel matrix $(a(m+n))$ acts boundedly on ℓ^2, then $a \in \ell^2$. The analytic *symbol* of this matrix is the L^2-function, g say, with $\hat{g}(n) = 0$ when $n < 0$ and with $\hat{g}(n) = a(n)$ otherwise.

Theorem HN. *A semi-infinite Hankel matrix acts boundedly on ℓ^2 if and only if there is a bounded function whose coefficients match the coefficients of the analytic symbol at all nonnegative indices.*

Examples. For the matrix H, use e^{-it} times the odd function that maps t to $\pi - t$ when $0 \leq t \leq \pi$. For the matrix A_v, use Theorem R or Observation F.

Theorem FH. *A semi-infinite Hankel matrix acts boundedly on ℓ^2 if and only if the analytic symbol of that matrix has bounded mean oscillation.*

Nonexamples. When $a \geq 0$, it is easy to verify that the symbol of $(a(m+n))$ has bounded mean oscillation if and only if

$$\sup_{M>0} \left\{ \sum_{j=1}^{\infty} \left[\sum_{jM \leq n < (j+1)M} a(n) \right]^2 \right\} < \infty.$$

Problem 1. Given such a sequence a build a sequence w for which the Schur test applies to the matrix $(a(m+n))$.

Problem 2. Given such a sequence a with finite support, apply the Schur algorithm to a suitably rescaled copy of a, and show that the algorithm does not terminate.