Approximations of functions that are analytic in a strip

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Functions of exponential type

$\mathcal{A}(2\pi\eta)$: all entire functions such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with

$$|A(z)| \leq C_\varepsilon e^{(2\pi\eta+\varepsilon)|z|}$$

for all $z \in \mathbb{C}$.

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In other words, supp$(\widehat{A}) \subset [-\eta, \eta]$.

Problem

Let $f : \mathbb{R} \to \mathbb{R}$ be given. Find $A \in \mathcal{A}(2\pi\eta)$ so that

- $\displaystyle\int_{\mathbb{R}} (A - f)$ is minimal,
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Some History:

- A. Beurling and A. Selberg considered this problem (independently) for x_+^0,
- S.W. Graham and J.D. Vaaler: $x_+^0 e^{-\lambda x}$,
- additional results by J.J. Holt and Vaaler, E. Carneiro and Vaaler, L.

- $A(x) \geq x_+^0$ then $G(x) := 1/2(A(x-a) + A(b-x)) \geq \chi_{[a,b]}$, so

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$$= \int_{-1}^{1} \widehat{A_\sigma}(y)\widetilde{\mu}(\sigma + 2\pi i y)dy$$

where $\widetilde{\mu}(s)$ is the Laplace transform of μ. Important: Integration range is $[-1,1]$!

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- What is the function A of best approximation?

Why $\cosh(ax)^{-1}$ **?**

Let U_a be the class of functions defined by the following properties:

1. $f : \{z : |\Im z| < a\} \to \mathbb{C}$ such that $f(\mathbb{R}) \subseteq \mathbb{R}$,
2. f is holomorphic on $|\Im z| < a$,
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Akhiezer: The distance from $f \in U_a$ to $\mathcal{A}(\eta)$ with respect to L^∞-norm is bounded by

$$\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cosh((2n+1)a\eta)}.$$

Akhiezer's approach

Any $f \in U_a$ has representation

$$f(z) = \int_{-\infty}^{\infty} h(t) \cosh(a(x-t))^{-1} dt$$

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- Mimic this: One-sided approximations to functions in U_a are obtainable from one-sided L^1-approximations to $\cosh(ax)^{-1}$.

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$$\sum_{\ell} (\widehat{f}(\ell) - \widehat{F}(\ell)) e^{2\pi i n \alpha} = \sum_{n} (f(n+\alpha) - F(n+\alpha)) \geq 0$$

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$$(\widehat{f - F})(0) + \sum_{\ell \neq 0} \widehat{f}(\ell) e^{2\pi i \ell \alpha} = \sum_n (f(n+\alpha) - F(n+\alpha)) \ge 0$$

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- Have equality provided $F(n+\alpha) = f(n+\alpha)$, some α, all $n \in \mathbb{Z}$.
- Construction: Interpolate f and f' at $n+\alpha$ by a function in $\mathcal{A}(2\pi)$ and hope for the best (i.e., non-positive or non-negative difference).

Construction

Consider $f(x) = \cosh(x)^{-1}$. Design $F \in \mathcal{A}(2\pi)$ with

$$F(x) \leq \cosh(x)^{-1} \text{ on } \mathbb{R},$$

$$F(n + 1/2) = \cosh(n + 1/2)^{-1} \text{ for } n \in \mathbb{Z}.$$

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Try

$$F(x) := \sum_{n\in\mathbb{Z}} \Big(\cosh(n+1/2)^{-1} \frac{\cos^2 \pi x}{\pi^2 (x-n-\frac{1}{2})^2} + \frac{d}{dn}[\cosh(n+1/2)^{-1}] \frac{\cos^2 \pi x}{\pi^2 (x-n-\frac{1}{2})} \Big).$$

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$$= 2 \sum_{k=0}^{\infty} (-1)^k e^{(2k+1)x} \frac{\cos^2 \pi x}{\pi^2} \int_{-\infty}^{\infty} e^{-xt} \frac{te^{-t/2}}{1-e^{-t}} dt,$$

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and expand the integrand for $t < 0$ and $t > 0$, i.e., for $t > 0$,

$$\frac{te^{-t/2}}{1-e^{-t}} = t \sum_{n=0}^{\infty} e^{-(n+1/2)t}$$

Let $0 < x < 1/2$. Recall

$$F(x) = \sum_{n \in \mathbb{Z}} \cosh(n + 1/2)^{-1} \frac{\cos^2 \pi x}{\pi^2 (x - n - \frac{1}{2})^2} + \ldots$$

Expand

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Combining these in the range $-1/2 < x < 0$ leads to

$$F(x) - \cosh(x)^{-1} = -2\frac{\cos^2 \pi x}{\pi^2} \int_{-\infty}^{0} e^{-xt}\gamma(t)dt,$$

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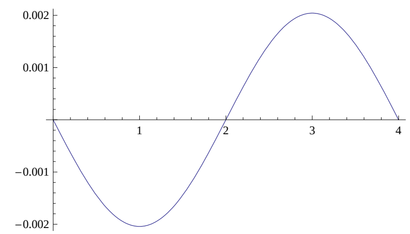
where γ is the 4-periodic function given by

$$\gamma(t) = \sum_{k=-\infty}^{\infty} (-1)^k g(t + 2k + 1)$$

with $g(t) = te^{t/2}(1 - e^t)^{-1}$.



$\gamma(t)$ in the range $0 \leq t \leq 4$

Define

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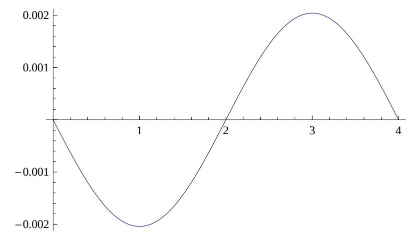

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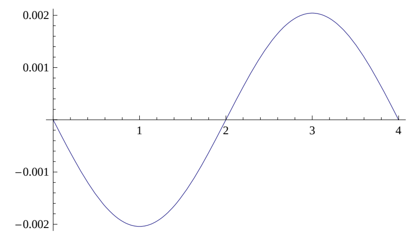

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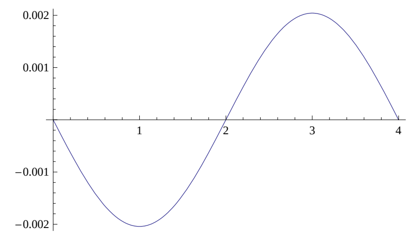

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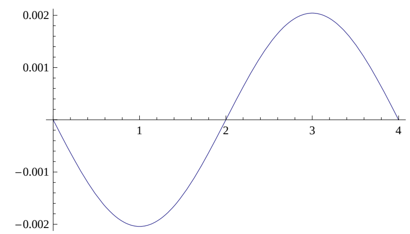

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- $F(x) - \cosh(x)^{-1} \le 0$ for all real x,
- $F(x) - \cosh(x)^{-1}$ is integrable on $\mathbb{R}$,
- $|F(x)| \ll (1 + |\cos \pi x|^2)$ in $|\Re x| \ge 1/4$.

The expansion $-1/2 < x < 0$

$$\cosh(x)^{-1} = 2\frac{\cos^2 \pi x}{\pi^2} \int_{-\infty}^{\infty} e^{-xt} \sum_{n=0}^{\infty} (-1)^n g(t + 2n + 1) dt,$$

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gives

$$F(x) = 2\frac{\cos^2 \pi x}{\pi^2} \Big[\int_0^{\infty} e^{-xt} \sum_{n=0}^{\infty} (-1)^n g(t+2n+1)dt - \int_{-\infty}^{0} e^{-xt} \sum_{n=-\infty}^{-1} (-1)^n g(t+2n+1)dt \Big],$$

which is valid in $-1/2 < x < 1/2$.

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- $F(z)$ is entire,
- $|F(z)| \leq C(1 + |\cos^2 \pi z|)$ for all $z \in \mathbb{C}$, hence by the Paley-Wiener theorem supp$(\widehat{F}) \subset [-1, 1]$.

Approximate $\cosh(ax)^{-1}$

The expansion

$$\cosh(ax)^{-1} = 2e^{ax}(1+e^{2ax})^{-1}$$
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Integral value (Poisson summation, Paley-Wiener, and interpolation at $n + 1/2$):

$$\int_{-\infty}^{\infty} (F_a(x) - \cosh(ax)^{-1}) dx = \frac{\pi}{a} \sum_{\ell \neq 0} \frac{(-1)^\ell}{\cosh(\frac{\pi^2 \ell}{a})}$$

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- Technical difficulty: Zero at the origin, hence g will grow polynomially as $t \to \infty$ or $t \to -\infty$ (depending on starting interval)

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Obtain $A \in \mathcal{A}(2\pi)$ with $A \geq \cosh(ax)^{-1}$ such that

$$\int (A(x) - \cosh(ax)^{-1}) dx = \frac{\pi}{a} \sum_{n \neq 0} \frac{1}{\cosh(\frac{\pi^2 n}{a})}$$