Surprising Sinc Sums and Integrals

David Borwein

University of Western Ontario

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1 Motivation and preliminaries.

This talk is based on material in a paper to appear shortly in MAA MONTHLY with the above title, co-authored with Robert Baillie and Jonathan M. Borwein. We show that a variety of trigonometric sums have unexpected closed forms by relating them to cognate integrals. We hope this offers a good advertisement for the possibilities of experimental mathematics, as well as providing both some entertaining examples for the classroom and a caution against over-extrapolating from seemingly compelling initial patterns.

Recall the standard convention $\operatorname{sinc}(x) := \sin(x)/x$ when $x \neq 0$ and $\operatorname{sinc}(0) := 1$. It is known that

$$\int_0^\infty \operatorname{sinc}(x)\,dx = \int_0^\infty \operatorname{sinc}^2(x)\,dx = \frac{\pi}{2}, \tag{1}$$

while

$$\sum_{n=1}^\infty \operatorname{sinc}(n) = \sum_{n=1}^\infty \operatorname{sinc}^2(n) = \frac{\pi}{2} - \frac{1}{2}. \tag{2}$$

Since sinc is an even function the mysterious $-1/2$ can be removed from (2) to get the equivalent statement

$$\begin{aligned}\int_{-\infty}^\infty \operatorname{sinc}(x)\,dx &= \int_{-\infty}^\infty \operatorname{sinc}^2(x)\,dx \\ &= \sum_{n=-\infty}^\infty \operatorname{sinc}(n) = \sum_{n=-\infty}^\infty \operatorname{sinc}^2(n) = \pi.\end{aligned}$$

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In the rest of the talk this sort of equivalence will not be restated and the more familiar one-sided sums and integrals will be used, rather than the two-sided versions which are more natural from a Fourier analysis perspective.

Experimentation with *Mathematica* suggested that for $N = 1, 2, 3, 4, 5$, and 6, the sum

$$\sum_{n=1}^{\infty} \operatorname{sinc}^N(n)$$

is $-1/2$ plus a rational multiple of π.

But for $N = 7$ and $N = 8$, the results are completely different: *Mathematica* gives polynomials in π of degree 7 and 8 respectively. For example, for $N = 7$, we get

$$-\frac{1}{2} + \frac{1}{46080}(129423\pi - 201684\pi^2 + 144060\pi^3 - 54880\pi^4 + 11760\pi^5 - 1344\pi^6 + 64\pi^7).$$

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These results are surprising, and we explain them below. But there's more.

Further experimentation suggested that for $N = 1, 2, 3, 4, 5$, and 6, (but not 7 or 8), we had

$$\sum_{n=1}^{\infty} \operatorname{sinc}^N(n) = -\frac{1}{2} + \int_0^{\infty} \operatorname{sinc}^N(x)\, dx\,. \tag{3}$$

This too was unexpected. In the integral test for infinite series, the convergence of the integral of $f(x)$ may imply the convergence of the sum of $f(n)$, but there is usually no simple relationship between the values of the sum and the corresponding integral.

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We found more examples where the sum was 1/2 less than the corresponding integral. In previous papers it was shown that, for $N = 0, 1, 2, 3, 4, 5,$ and 6,

$$\int_0^\infty \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{2k+1}\right) dx = \frac{\pi}{2}, \tag{4}$$

but that for $N = 7$, the integral is just slightly less than $\pi/2$:

$$\int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{15}\right) dx$$
$$= \pi\left(\frac{1}{2} - \frac{6879714958723010531}{935615849440640907310521750000}\right).$$

This surprising sequence is explained by a result in a previous paper which is incorporated into Theorem 2 below.

More experiments suggested that, for $N = 0, 1, 2, 3, 4, 5, 6, \textit{and}\ 7$, the sums were also 1/2 less than the corresponding integrals:

$$\sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{n}{2k+1}\right) = -\frac{1}{2} + \int_{0}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{2k+1}\right) dx\,. \quad (5)$$

In fact, we show in Example 1 (b) below that (5) holds for every $N \leq 40248$ and fails for *all* larger integers! This certainly underscores the need for caution, mentioned above.

We now turn to showing that the theorems for integrals proven elsewhere imply analogues for sums. Our results below use basic Fourier analysis, all of which can be found in standard texts, to explain the above sums, and others, and to allow us to express many such sums in closed form.

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2 When sums and integrals agree.

Suppose that G is Lebesgue integrable over $(-\infty, \infty)$ and define its Fourier transform g by

$$g(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iux} G(u)\, du.$$

At any point u such that G is of bounded variation on $[u-\delta, u+\delta]$ for some $\delta > 0$ we have by a standard result that

$$\frac{1}{2}\{G(u+) + G(u-)\} = \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} e^{iux} g(x)\, dx, \qquad (6)$$

where $G(u\pm)$ denotes $\lim_{x\to u^{\pm}} G(x)$.

Suppose, in addition, that $G(x) = 0$ for $x \notin (-\alpha, \alpha)$ for some $\alpha > 0$, and that G is of bounded variation on $[-\delta, \delta]$ for some $\delta > 0$. Then clearly

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{-iux} G(u)\, du,$$

and hence, for $r = 0, 1, 2, \ldots$, by summing the exponential,

$$\sum_{n=-r}^{r} g(n) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} G(u) \frac{\sin((r + 1/2)u)}{\sin(u/2)} du. \qquad (7)$$

Suppose first that $0 < \alpha < 2\pi$. Then

$$\sum_{n=-r}^{r} g(n) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} G^*(u) \frac{\sin((r+1/2)u)}{u} du, \tag{8}$$

where

$$G^*(u) := G(u) \frac{u}{\sin(u/2)}. \tag{9}$$

Since G^* is of bounded variation on $[-\delta, \delta]$ and Lebesgue integrable over $(-\alpha, \alpha)$, and $G^*(0+) = 2G(0+)$ and $G^*(0-) = 2G(0-)$, it follows, by a standard Jordan-type result, that

$$\lim_{r\to\infty} \int_{-\alpha}^{\alpha} G^*(u) \frac{\sin((r+1/2)u)}{u} du$$
$$= \frac{\pi}{2}\{G^*(0+) + G^*(0-)\} = \pi\{G(0+) + G(0-)\}. \tag{10}$$

The following proposition, which enables us to explain most of the above experimental identities, now follows from (6) with $u = 0$, (8), (9), and (10).

Proposition 1

If G is of bounded variation on $[-\delta, \delta]$, vanishes outside $(-\alpha, \alpha)$, and is Lebesgue integrable over $(-\alpha, \alpha)$ with $0 < \alpha < 2\pi$, then

$$\lim_{r\to\infty} \sum_{n=-r}^{r} g(n) = \lim_{T\to\infty} \int_{-T}^{T} g(x)\,dx = \sqrt{\frac{\pi}{2}}\{G(0+) + G(0-)\}. \quad (11)$$

As a simple illustration, consider the function G that equals 1 in the interval $(-1, 1)$ and 0 outside. The corresponding g is given by $g(x) = \sqrt{2/\pi}\,\mathrm{sinc}(x)$. Then (11) shows, since $\mathrm{sinc}(x)$ is an even function, that

$$1 + 2\sum_{n=1}^{\infty} \mathrm{sinc}(n) = 2\int_0^{\infty} \mathrm{sinc}(x)\,dx = \pi,$$

where the integral is an improper Riemann integral.

The prior analysis can be taken further, assuming only that $G(x) = 0$ for $x \notin (-\alpha, \alpha)$ for some $\alpha > 0$. Suppose first that $2\pi \leq \alpha < 4\pi$ and that G is also of bounded variation on $[-2\pi - \delta, -2\pi + \delta]$ and $[2\pi - \delta, 2\pi + \delta]$. Then, by splitting the integral in (7) into three parts and making the appropriate changes of variables, we get

$$\begin{aligned}\sum_{n=-r}^{r} g(n) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} G(u) \frac{\sin((r+1/2)u)}{\sin(u/2)} du \\ &+ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\alpha-2\pi} G(u+2\pi) \frac{\sin((r+1/2)u)}{\sin(u/2)} du \\ &+ \frac{1}{\sqrt{2\pi}} \int_{2\pi-\alpha}^{\pi} G(u-2\pi) \frac{\sin((r+1/2)u)}{\sin(u/2)} du.\end{aligned}$$

Hence, from this we get as in the previous case that when $\alpha = 2\pi$,

$$\lim_{r\to\infty} \sum_{n=-r}^{r} g(n) = \lim_{T\to\infty} \int_{-T}^{T} g(x)\,dx + \sqrt{\frac{\pi}{2}}\{G(2\pi-) + G(-2\pi+)\}, \tag{12}$$

and when $2\pi < \alpha < 4\pi$,

$$\lim_{r\to\infty} \sum_{n=-r}^{r} g(n) = \lim_{T\to\infty} \int_{-T}^{T} g(x)\,dx + \sqrt{\frac{\pi}{2}}\{G(2\pi-) + G(2\pi+) + G(-2\pi-) + G(-2\pi+)\}. \tag{13}$$

This process can evidently be continued by induction to yield that, when $2m\pi < \alpha < 2(m+1)\pi$ with m a positive integer, and G is of bounded variation in intervals containing the points $\pm 2n\pi,\ n = 0, 1, \ldots, m$,

$$\lim_{r\to\infty} \sum_{n=-r}^{r} g(n) = \lim_{T\to\infty} \int_{-T}^{T} g(x)\,dx + \sqrt{\frac{\pi}{2}} R_m, \tag{14}$$

where $R_0 := 0$ and for $k > 0$,

$$R_k := \sum_{n=1}^{k} \{G(2n\pi-) + G(2n\pi+) + G(-2n\pi-) + G(-2n\pi+)\}.$$

Correspondingly, when $\alpha = 2m\pi$,

$$\lim_{r\to\infty} \sum_{n=-r}^{r} g(n) = \lim_{T\to\infty} \int_{-T}^{T} g(x)\,dx + \sqrt{\frac{\pi}{2}} \{R_{m-1} + G(2m\pi-) + G(-2m\pi+)\}. \tag{15}$$

3 Applications to sinc sums.

For an application of the above analysis let

$$g(x) := \prod_{k=0}^{N} \operatorname{sinc}(a_k x)$$

with all $a_k > 0$, and let

$$A_N := \sum_{k=0}^{N} a_k.$$

It has been shown elsewhere that the corresponding Fourier transform G is positive and continuous in the interval $I_N := (-A_N, A_N)$ and 0 outside the closure of I_N, and is of bounded variation on every finite interval; indeed, G is known to be absolutely continuous on $(-\infty, \infty)$ when $N \geq 1$.

It therefore follows from (11) along with (12) that

$$1 + 2\sum_{n=1}^{\infty}\prod_{k=0}^{N}\operatorname{sinc}(a_k n) = 2\int_0^{\infty}\prod_{k=0}^{N}\operatorname{sinc}(a_k x)\,dx, \tag{16}$$

provided

$$A_N \leq 2\pi \text{ when } N \geq 1, \text{ or } A_N < 2\pi \text{ when } N = 0. \tag{17}$$

The proviso is needed since (14) and (15) tell us that the left-hand side of (16) is strictly greater than right-hand side when (17) doesn't hold, since then either (i) $N \geq 1$, $A_N > 2\pi$, and $G(\pm 2\pi) > 0$ or (ii) $N = 0$, $A_N \geq 2\pi$, and $G(2\pi-) + G(-2\pi+) > 0$, and in either case all other terms that comprise the remainder R_N are non-negative.

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We can go further and say that when the proviso fails the constant 1 on the left-hand side of (16) has to be replaced by a constant $C < 1$ that depends only on the value of A_N; unfortunately there appears to be no neat expression for C. We emphasize that though the case $N = 0$ follows from the above analysis, neither the series nor the integral in (16) is absolutely convergent in this case. For all other values of N both are absolutely convergent.

Though this “sum=integral" paradigm is very general. there are not too many “natural” analytic g for which G is as required—other than powers and other relatives of the sinc function. A nice example first found by Shisha and Pollard obtained by taking $G(t) := \left(1+e^{it}\right)^{\alpha}$ for $|t| \le \pi$ and zero otherwise is:

$$\sum_{n=-\infty}^{\infty} \binom{\alpha}{n} e^{int} = \int_{-\infty}^{\infty} \binom{\alpha}{u} e^{itu}\, du = \left(1+e^{it}\right)^{\alpha}$$

for $\alpha > -1, |t| < \pi$.

Additionally, however, in the case of sinc integrals, as explained elsewhere, the right-hand term in (16) is equal to $2^{-N} V_N \pi / a_0$, where V_N is the—necessarily rational when the a_k are—volume of the part of the cube $[-1,1]^N$ between the parallel hyperplanes

$a_1 x_1 + a_2 x_2 + \cdots + a_N x_N = -a_0$ and $a_1 x_1 + a_2 x_2 + \cdots + a_N x_N = a_0$.

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$$a_1x_1+a_2x_2+\cdots+a_Nx_N = -a_0 \text{ and } a_1x_1+a_2x_2+\cdots+a_Nx_N = a_0.$$

Theorem 1 (Sinc Sums)

One has

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(a_k n) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(a_k x)\, dx = \frac{\pi}{2a_0} \frac{V_N}{2^N} \leq \frac{\pi}{2a_0} \tag{18}$$

where the first equality holds provided

$$A_N = \sum_{k=0}^{N} a_k \leq 2\pi \text{ when } N \geq 1, \text{ or } A_N < 2\pi \text{ when } N = 0. \tag{19}$$

The second equality needs no such restriction. Moreover (18) holds with equality throughout provided additionally that

$$A_N < 2\, a_0. \tag{20}$$

Various extensions are possible when (19) or (20) fail. The following corollary follows immediately from (18) on making the substitution $x = \tau t$ in the integral.

Corollary 1

Let τ be any positive number such that $0 < \tau A_N \leq 2\pi$ when $N \geq 1$, or $0 < \tau A_N < 2\pi$ when $N = 0$. Then

$$\begin{aligned} \frac{\tau}{2} + \tau \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(\tau a_k n) &= \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(a_k x)\, dx \\ &= \frac{\pi}{2a_0} \frac{V_N}{2^N} \leq \frac{\pi}{2a_0}. \end{aligned} \tag{21}$$

In particular, (21) is independent of τ in the given interval.

When (20) fails but $A_{N-1} < 2\,a_0$, as proven elsewhere we may specify the volume change:

Theorem 2 (First Bite)

Suppose that $2a_k \geq a_N$ for $0 \leq k \leq N-1$ and that $A_{N-1} \leq 2a_0 < A_N$, and $0 < \tau\, A_N \leq 2\pi$. Then, for $0 \leq r \leq N-1$,

$$\frac{\tau}{2} + \tau \sum_{n=1}^{\infty} \prod_{k=0}^{r} \operatorname{sinc}(\tau a_k n) = \int_0^{\infty} \prod_{k=0}^{r} \operatorname{sinc}(a_k x)\, dx = \frac{\pi}{2a_0}, \quad (22)$$

while

$$\frac{\tau}{2} + \tau \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(\tau a_k n) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(a_k x)\, dx = \frac{\pi}{2a_0}\left(1 - \frac{(A_N - 2a_0)^N}{2^{N-1} N! \prod_{k=1}^{N} a_k}\right). \quad (23)$$

4 Examples and extensions.

We may now explain the original discoveries:

Example 1

(a) Let N be an integer and for $k = 0, 1, \ldots, N$, let $a_k := 1/(2k+1)$. If N is in the range $1 \leq N \leq 6$, then

$$A_N = \sum_{k=0}^{N} a_k < 2a_0 \text{ and } A_N < 2\pi.$$

Hence, for each of these N, conditions (19) and (20) of Theorem 1 hold and so we can apply that theorem to get

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{n}{2k+1}\right) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{2k+1}\right) dx = \frac{\pi}{2}.$$

Now for $N = 7$, condition (20) fails because

$$\begin{aligned} A_N &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} > 2a_0 = 2 \\ &> A_{N-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13}. \end{aligned}$$

However, the conditions of Theorem 2 are met, namely

$$A_{N-1} = \frac{88069}{45045} \leq 2a_0 < A_N = \frac{91072}{45045} < 2\pi,$$

and for each $k = 0, 1, \ldots, N-1$, we have $2a_k > a_N$.

Therefore, we can take $\tau = 1$ and apply equation (23) of Theorem 2 to get

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{7} \operatorname{sinc}\left(\frac{n}{2k+1}\right) = \int_0^\infty \prod_{k=0}^{7} \operatorname{sinc}\left(\frac{x}{2k+1}\right) dx$$

$$= \frac{\pi}{2}\left(1 - \frac{\left(\frac{91072}{45045} - 2\right)^7}{2^6 7! \cdot \frac{1}{3} \cdot \frac{1}{5} \cdots \frac{1}{15}}\right)$$

$$= \frac{\pi}{2}\left(1 - \frac{6879714958723010531}{467807924720320453655260875000}\right).$$

(b) Let a_k be as in part (a). If $7 \leq N \leq 40248$, then $\sum_{k=0}^{N} a_k < 2\pi$, so (19) holds but (20) does not. For each of these N, Theorem 1 tells us that

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{n}{2k+1}\right) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{2k+1}\right) dx < \frac{\pi}{2}.$$

For $N > 40248$, the equality in the above formula fails. Indeed, equation (14) shows that

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{n}{2k+1}\right) > \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{2k+1}\right) dx,$$

since the error term is necessarily strictly positive for the requisite G, which was discussed at the beginning of the previous section. In a remarkable analysis based on random walks, Crandall rigorously estimates that the error for $N = 40249$ is minuscule: less than $10^{-226576}$. The integral is still less than $\pi/2$. Moreover, all subsequent errors are provably no larger than 10^{-13679}.

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(c) Let $a_k := 1/(k+1)^2$. Because $\sum_{k=0}^{\infty} a_k$ converges with sum $\pi^2/6$ which is both less than 2π and less than $2a_0 = 2$, Theorem 1 says that, for *every* $N \geq 0$,

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{n}{(k+1)^2}\right) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{(k+1)^2}\right) dx = \frac{\pi}{2}.$$

Thus, no matter how many factors we include, both sum and integral are unchanged! In fact, if the a_k are the terms of any positive infinite series that converges to a sum less than $\min(2\pi, 2a_0)$, then for *every* $N \geq 0$,

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(a_k n) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(a_k x)\, dx = \frac{\pi}{2}.$$

Example 2

(a) Let $a_0 := 1$ and—to inject a little number theory—let $a_1, a_2, \ldots, a_9$ be the reciprocals of the odd primes $3, 5, 7, 11, \ldots, 29$. Then the conditions of Theorem 2 are satisfied, so

$$\begin{aligned}
\frac{1}{2} &+ \sum_{n=1}^{\infty} \operatorname{sinc}(n)\operatorname{sinc}(n/3)\cdots\operatorname{sinc}(n/23)\operatorname{sinc}(n/29) \\
&= \int_0^{\infty} \operatorname{sinc}(x)\operatorname{sinc}(x/3)\cdots\operatorname{sinc}(x/23)\operatorname{sinc}(x/29)\,dx \\
&= \pi\left(\frac{1}{2} - \frac{3959735161333055430365 0 8\cdots}{43510349833593819674958681335\cdots}\right) \\
&\sim 0.499999990899\,\pi.
\end{aligned}$$

(b) In the same vein, if we tweak the sequence very slightly by taking the reciprocals of all the primes (i.e., the first term is $1/2$ not 1), then we have the 'sum plus $1/2$' and the integral equalling π only for $N = 0$ or 1. For $N = 2$, Theorem 2 tells us each equals $\pi(1 - 1/240)$. However, the first equality in Theorem 1 holds until $\sum_{k=0}^{N} a_k$ exceeds 2π.

We now estimate the N for which this occurs. The sum of the reciprocals of the primes diverges slowly. In fact, $\sum\{1/p \colon p \le x, p \,\mathrm{prime}\}$ is roughly $\log(\log(x)) + B$, where $B \sim 0.26149\ldots$ is the *Mertens constant*.

In order for this sum to exceed 2π, x must be about $y = \exp(\exp(2\pi - B)) \sim 10^{179}$. Thus, by the Prime Number Theorem, $N \sim y/\log(y)$, which is about 10^{176}. Thus, anyone who merely tested examples using these a_k would almost certainly *never* find an integer N where the first equality in Theorem 1 failed.

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This time Crandall proves—unconditionally—the consequent error to be less than $10^{-\left(10^{86}\right)}$. Moreover, assuming the Riemann hypothesis this upper bound reduces to $10^{-\left(10^{176}\right)}$, which is much less than one part in a googolplex.

Example 3

Let $1 \leq N \leq 6$, and take $a_0 = a_1 = \cdots = a_{N-1} = 1$. Then condition (19) with N replaced by $N-1$ is satisfied, so equation (18) of Theorem 1 tells us that for each $N = 1, 2, 3, 4, 5$, and 6, we have

$$\sum_{n=1}^{\infty} \operatorname{sinc}^N(n) = -\frac{1}{2} + \int_0^{\infty} \operatorname{sinc}^N(x)\, dx.$$

Moreover, for each $N \geq 1$ the integral is an effectively computable rational multiple of π, the numerator and denominator of which are listed by Sloane. If $N = 7$, then $A_{N-1} = 7 > 2\pi$, so (19) with N replaced by $N-1$ is no longer satisfied and, in this case, as Example 4 shows, the sum and the integral do not differ by 1/2. Indeed, for $N \geq 7$, the sums have an entirely different quality: they are polynomials in π of degree N.

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We continue this discussion in the next counterexample, for which we define, for $N = 1, 2, \ldots,$

$$i_N := \int_0^\infty \operatorname{sinc}^N(x)\, dx, \qquad s_N := \sum_{n=1}^\infty \operatorname{sinc}^N(n).$$

Example 4

(a) We saw in Example 3 that for $N = 1, 2, 3, 4, 5,$ and 6, we have $s_N = i_N - 1/2$. By contrast $i_7 = 5887\pi/23040$, but *Mathematica 6* gives

$$s_7 = -\frac{1}{2} + \frac{43141}{15360}\pi - \frac{16807}{3840}\pi^2 + \frac{2401}{768}\pi^3 - \frac{343}{288}\pi^4 + \frac{49}{192}\pi^5 - \frac{7}{240}\pi^6 + \frac{1}{720}\pi^7. \tag{24}$$

Similarly, $i_8 = 151\pi/360$, and *Mathematica 6* gives

$$s_8 = -\frac{1}{2} + \frac{733\pi}{210} - \frac{256\pi^2}{45} + \frac{64\pi^3}{15} - \frac{16\pi^4}{9} + \frac{4\pi^5}{9} - \frac{\pi^6}{15} + \frac{\pi^7}{180} - \frac{\pi^8}{5040}. \quad (25)$$

Although (19) fails, we can explain these sums, and we will show how to express s_N in closed form.

(b) For $N \leq 6$, s_N is 1/2 less than a rational multiple of π. The sudden change to a polynomial in π of degree N is explained by the use of trigonometric identities and known Bernoulli polynomial evaluations of Fourier series. In general, we have the following two identities:

$$\sin^{2N+1}(n) = \frac{1}{2^{2N}} \sum_{k=1}^{N+1} (-1)^{k+1} \binom{2N+1}{N-k+1} \sin\left((2k-1)n\right) \quad (26)$$

and

$$\sin^{2N}(n) = \frac{1}{2^{2N-1}} \left(\frac{1}{2} \binom{2N}{N} + \sum_{k=1}^{N} (-1)^k \binom{2N}{N-k} \cos(2kn) \right). \quad (27)$$

In particular, to compute s_7, we start with

$$\sin^7(n) = \frac{35}{64}\sin(n) - \frac{21}{64}\sin(3\,n) + \frac{7}{64}\sin(5\,n) - \frac{1}{64}\sin(7\,n)\,. \tag{28}$$

Now, for $0 \le x \le 2\pi$,

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{2N+1}} = \frac{(-1)^{N-1}}{2}(2\pi)^{2N+1}\,\phi_{2N+1}\left(\frac{x}{2\pi}\right) \tag{29}$$

and

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{2N}} = \frac{(-1)^{N-1}}{2}(2\pi)^{2N}\phi_{2N}\left(\frac{x}{2\pi}\right), \tag{30}$$

where $\phi_N(x)$ is the Nth *Bernoulli polynomial*, normalized so that the high-order coefficient is $1/N!$.

We divide (28) by n^7 and sum over n. Then, we would like to use (29) four times with $N = 3$ and $x = 1, 3, 5, 7$. But there is a hitch: (29) is not valid for $x = 7$ because $x > 2\pi$. So instead of 7 we use $7 - 2\pi$. It is this value, $7 - 2\pi$, substituted into the Bernoulli polynomial, that causes s_7 to be a 7th degree polynomial in π. For s_{13}, for example, we would have to use $x = 1, 3, 5, 7 - 2\pi, 9 - 2\pi, 11 - 2\pi$, and $13 - 4\pi$. For $N \geq 7$, we would end up with an Nth degree polynomial in π.

(c) With more effort this process yields a closed form for each such sum. First, for $N = 7$ we have observed that

$$-64\sin^7(n) = \sin(7n) - 7\sin(5n) + 21\sin(3n) - 35\sin(n), \quad (31)$$

and that

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^7} = 64\pi^7 \phi_7\left(\frac{x}{2\pi}\right) \text{ for } 0 \le x \le 2\pi, \quad (32)$$

where

$$\phi_7(x) := \frac{1}{30240}x - \frac{1}{4320}x^3 + \frac{1}{1440}x^5 + \frac{1}{5040}x^7 - \frac{1}{1440}x^6 \quad (33)$$

is the Bernoulli polynomial of order seven. Note that in (31), 7 is the only coefficient that falls outside the interval $(0, 2\pi)$.

Substituting (32) into (31) yields (24), provided instead of simply replacing x with 7, we replace x with $7 - 2\pi$ when dealing with the $\sin(7n)$ term in (31), to stay in the interval where (32) is valid. The same procedure, with versions of (32) and (31) using cosines in place of sines, yields (25). An interesting additional computation shows that

$$s_7 + \frac{1}{2} - i_7 = 64\pi^7 \left\{ \phi_7\left(\frac{7-2\pi}{2\pi}\right) - \phi_7\left(\frac{7}{2\pi}\right) \right\}. \qquad (34)$$

In other words the difference between $s_7 + 1/2$ and i_7 resides in the one term in (31) with coefficient outside the interval $(0, 2\pi)$.

(d) Let us use the fractional part

$$\{z\}_{2\pi} := \frac{z}{2\pi} - \left\lfloor \frac{z}{2\pi} \right\rfloor .$$

In like fashion, we ultimately obtain pretty closed forms for each s_M.

For M odd:

$$s_M = \frac{(-1)^{\frac{M+1}{2}}}{M!}\pi^M \sum_{k=1}^{\frac{M+1}{2}} (-1)^{k+1} \binom{M}{\frac{M+1}{2}-k} \phi_M\left(\{2k-1\}_{2\pi}\right).$$

For M even:

$$s_M = \frac{(-1)^{M/2}}{M!}\pi^M \sum_{k=0}^{\frac{M}{2}} \frac{(-1)^{k+1}}{\delta_{k,0}+1} \binom{M}{\frac{M}{2}-k} \phi_M\left(\{2k\}_{2\pi}\right),$$

where, as usual, $\delta_{k,0} = 1$ when $k = 0$, and 0 otherwise.
Remarkably, these formulae are rational multiples of π exactly for $M \leq 6$ and thereafter are polynomials in π of degree M.

Many variations on the previous themes are possible. In simple cases it is easy to proceed as follows:

Example 5

Let us introduce the notation

$$s_{i,j} := \sum_{n=1}^{\infty} \operatorname{sinc}(n)^i \cos(n)^j.$$

We discovered experimentally that $s_{1,1} = s_{1,2} = s_{2,1} = s_{3,1} = s_{2,2} = \pi/4 - 1/2$ and that in each case the corresponding integral equals $\pi/4$. Likewise $s_{1,3} = s_{2,3} = s_{3,3} = s_{1,4} = s_{2,4} = 3\pi/16 - 1/2$ while the corresponding integrals are equal to $3\pi/16$. Except for $s_{1,2}, s_{2,2}, s_{2,3}$, and $s_{1,4}$, the identity $\operatorname{sinc}(n)\cos(n) = \operatorname{sinc}(2n)$ allows us to apply Theorem 1. In the remaining four cases, we may use the method of Example 4 to prove the discovered results, but a good explanation has eluded us.

5 An extremal property.

We finish with a useful Siegel-type lower bound, giving an extremal property of the sinc^k integrals. This has applications to giving an upper bound on the size of integral solutions to integer linear equations.

Theorem 3 (Lower Bound)

Suppose $a_0 \geq a_k > 0$ for $k = 1, 2, \ldots, n$. Then

$$\int_0^\infty \prod_{k=0}^{n} \mathrm{sinc}(a_k x)\, dx \geq \int_0^\infty \mathrm{sinc}^{n+1}(a_0 x)\, dx. \tag{35}$$

In view of Corollary 1 we then have the following:

Corollary 2

Suppose $a_0 \geq a_k > 0$ for $k = 1, 2, \ldots, n$ and $0 < \tau A_n < 2\pi$. Then

$$\frac{\tau}{2} + \tau \sum_{r=1}^{\infty} \prod_{k=0}^{n} \operatorname{sinc}(\tau a_k r) = \int_0^{\infty} \prod_{k=0}^{n} \operatorname{sinc}(a_k x)\, dx \geq \int_0^{\infty} \operatorname{sinc}^{n+1}(a_0 x)\, dx. \quad (36)$$

Proof of Theorem 3.

Let

$$\tau_n := \int_0^\infty \prod_{k=0}^n \operatorname{sinc}(a_k x)\,dx, \quad \mu_n := \int_0^\infty \operatorname{sinc}^{n+1}(a_0 x)\,dx,$$

and, for $a > 0$, let

$$\chi_a(x) := \begin{cases} 1, & \text{if } |x| < a; \\ 1/2, & \text{if } |x| = a; \\ 0, & \text{if } |x| > a. \end{cases}$$

Further, let

$$F_0 := \frac{1}{a_0}\sqrt{\frac{\pi}{2}}\chi_{a_0}, \quad F_n := (\sqrt{2\pi})^{1-n} f_1 * f_2 * \cdots * f_n,$$

where

$$f_n := \frac{1}{a_n}\sqrt{\frac{\pi}{2}}\chi_{a_n},$$

and * indicates convolution, i.e.,

$$f_j * f_k(x) := \int_{-\infty}^{\infty} f_j(x-t) f_k(t)\, dt.$$

Then F_0 is the Fourier transform of $\operatorname{sinc}(a_0 x)$ and, for $n \geq 1$, F_n is the Fourier transform of $\prod_{k=1}^{n} \operatorname{sinc}(a_k x)$.

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Then F_0 is the Fourier transform of $\mathrm{sinc}(a_0 x)$ and, for $n \geq 1$, F_n is the Fourier transform of $\prod_{k=1}^{n} \mathrm{sinc}(a_k x)$.

In addition, for $n \geq 1$, $F_n(x)$ is an even function which vanishes on $(-\infty, -\sigma_n] \cup [\sigma_n, \infty)$ and is positive on $(-\sigma_n, \sigma_n)$, where $\sigma_n := A_n - a_0 = a_1 + a_2 + \cdots + a_n$. Furthermore, for $n \geq 1$, $F_n(x)$ is monotone nonincreasing on $(0, \infty)$. Hence, by a version of Parseval's theorem ,

$$\tau_n = \int_0^\infty F_n(x) F_0(x)\, dx = \frac{1}{a_0}\sqrt{\frac{\pi}{2}} \int_0^{\min(\sigma_n, a_0)} F_n(x)\, dx \text{ for } n \geq 1. \quad (37)$$

Observe that, for $n \geq 2$,

$$F_n = \frac{1}{\sqrt{2\pi}} F_{n-1} * f_n,$$

and hence that, for $y > 0$,

$$\begin{aligned}
\int_0^y F_n(v)\,dv &= \frac{1}{\sqrt{2\pi}} \int_0^y dv \int_{-\infty}^{\infty} F_{n-1}(v-t) f_n(t)\,dt \\
&= \frac{1}{2a_n} \int_0^y dv \int_{-a_n}^{a_n} F_{n-1}(v-t)\,dt \\
&= \frac{1}{2a_n} \int_{-a_n}^{a_n} dt \int_0^y F_{n-1}(v-t)\,dv \\
&= \frac{1}{2a_n} \int_{-a_n}^{a_n} dt \int_{-t}^{y-t} F_{n-1}(u)\,du.
\end{aligned}$$

Thus, we determine that

$$\int_0^y F_n(v)\,dv = \int_0^y F_{n-1}(u)\,du + I_1(a_n) + I_2(a_n), \tag{38}$$

where, for $x > 0$,

$$I_1(x) := \frac{1}{2x}\int_{-x}^{x} dt \int_{-t}^{0} F_{n-1}(u)\,du$$

and

$$I_2(x) := \frac{1}{2x}\int_{-x}^{x} dt \int_{y}^{y-t} F_{n-1}(u)\,du.$$

Now $I_1(x) = 0$ since $\int_{-t}^{0} F_{n-1}(u)\,du$ is an odd function of t, and for $y \geq x$,

$$I_2(x) = \frac{1}{2x}\int_0^x dt \int_y^{y-t} F_{n-1}(u)\,du + \frac{1}{2x}\int_{-x}^0 dt \int_y^{y-t} F_{n-1}(u)\,du$$
$$= \frac{1}{2x}\int_0^x \phi(t)\,dt, \tag{39}$$

where

$$\phi(t) := \int_y^{y+t} F_{n-1}(u)\,du - \int_{y-t}^{y} F_{n-1}(u)\,du \leq 0 \text{ for } 0 \leq t \leq y, \tag{40}$$

since $F_{n-1}(u)$ is monotonic nonincreasing for $u \geq 0$.

Observe that $\phi'(t) = F_{n-1}(y+t) - F_{n-1}(y-t) \leq 0$ for $0 \leq t \leq y$, apart from at most two exceptional values of t when $n = 2$. Hence

$$I_2'(x) = \frac{1}{x^2} \int_0^x \left(\phi(x) - \phi(t)\right) dt = \frac{1}{x^2} \int_0^x dt \int_t^x \phi'(u)\, du \leq 0,$$

and so

$$I_2(x) \text{ is nonincreasing for } 0 \leq x \leq y. \tag{41}$$

Our aim is to prove that $\tau_n \geq \mu_n$. Since, by Theorem 1, this inequality automatically holds when $a_0 \geq \sigma_n$, we assume that $a_0 < \sigma_n$. Note that in case $n = 1$ the hypothesis $a_0 \geq a_1 = \sigma_1$ immediately implies the desired inequality.

Assume therefore that $n \geq 2$ in the rest of the proof. Suppose $a_0, a_1, \ldots, a_n$ are not all equal, and re-index them so that a_0 remains fixed and $a_n < a_{n-1} \leq a_0$. If a_n is increased to a_{n-1}, it follows from (41) with $x = a_n$ and $y = a_0$ that $I_2(a_n)$ is not increased and hence, by (37), and (38) with $y = a_0$, that τ_n is not increased. Continuing in this way, we can coalesce all the a_k's into the common value a_0 without increasing the value of τ_n. This final value of τ_n is, of course, μ_n, and so the original value of τ_n satisfies $\tau_n \geq \mu_n$, as desired. Ω

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