Neighbouring Powers

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Happy 55th Birthday, Peter

Let $x, y \in \mathbb{Z}^+$.

How small is $|y^2 - x^3|$?

Zero of course.

Assume $|y^2 - x^3| \neq 0$.

$$3^2 - 2^3 = 1.$$

1922 Mordell $\quad k \in \mathbb{Z},\ k \neq 0$

$$y^2 - x^3 = k,$$

only finitely many solutions in integers x, y.

Thus

$$|y^2 - x^3| \to \infty \quad \text{as } x \to \infty.$$

Stark 1973

for each $\varepsilon > 0 \; \exists \; C(\varepsilon) > 0$ such that

$$|y^2 - x^3| > C(\varepsilon)(\log x)^{1-\varepsilon}.$$

Hall 1971 (Conj)

$\exists \; C > 0$ such that

$$|y^2 - x^3| > Cx^{1/2}$$

Likely Hall's conjecture should be modified

$$Cx^{1/2} \quad \text{should be} \quad C(\varepsilon)x^{1/2-\varepsilon},$$

for each $\varepsilon > 0$.

How does one produce examples $x, y \in \mathbb{Z}^+$ for which $|y^2 - x^3|$ is small?

Strategy: Find polynomials with coefficients in $\mathbb{Z}$ or $\mathbb{Q}$, g and f of degrees $3m$ and $2m$ respectively such that

$$g(t)^2 - f(t)^3$$

is a non-zero polynomial of small degree, say d.

By specializing to $t \in \mathbb{Z}^+$ we find

$$|y^2 - x^3| < c_1 x^{\frac{3d}{6m}} = c_1 x^{\frac{d}{2m}}.$$

1961 Birch: an example with $m = 5$ and $d = m + 1$.

$$g(t) = \frac{t^{15}}{27} + \frac{t^{12} + 4t^9 + 8t^6}{3} + \frac{5t^3 + 1}{2},$$

$$f(t) = \frac{t}{9}(t^9 + 6t^6 + 15t^3 + 12)$$

1965 Birch, Chowla, Hall and Schinzel

$$0 < |y^2 - x^3| < c_1 x^{3/5}$$

for infinitely many pairs (x, y).

Ideally we want d small and m large but there are limits.

Davenport 1965

$$d \geq m + 1.$$

If we find f and g for which $d = m + 1$ then

$$0 < |y^2 - x^3| < c_1 x^{\frac{1}{2} + \frac{1}{2m}}$$

for infinitely many (x, y).

No example with $m > 5$ known.

Difficulty: to find $f, g \in \mathbb{Q}[t]$.

1982 Danilov

$$(t^2 + 10t + 5)^3 - (t^2 + 22t + 125)(t^2 + 4t - 1)^2 = 1728t.$$

He replaced t by $5^3 T$ and factored through by 5^3.

Taking $T = -11$, $5^3 T^2 + 22T + 1$ becomes a square (Pell's equation).

Thus

$$0 < |y^2 - x^3| < cx^{1/2} \quad \text{for } c > 0$$

and ∞-ly many (x, y).

Mason and Stothers

Let $f, g, h \in \mathbb{C}[x]$.

Suppose f, g, h are pairwise coprime and

$$f + g = h.$$

Then

$$\max(\deg f, \deg g, \deg h) \leq N_0(fgh) - 1,$$

where for any polynomial F, $N_0(F)$ denotes the number of distinct zeros of F.

The *abc* conjecture:

Let $x, y, z \in \mathbb{Z}^+$ and define $G = G(x, y, z)$ by

$$G = \prod_{p \mid xyz} p.$$

Suppose $(x, y, z) = 1$ and $x + y = z$.

(Oesterlé-Masser) Conjecture 1985.

For each $\varepsilon > 0 \ \exists \ C_1(\varepsilon) > 0$ such that

$$z < C_1(\varepsilon) G^{1+\varepsilon}.$$

Let n and m be coprime integers with $n > m \geq 2$.

How small is $|x^n - y^m|$ when x and y are positive integers and $x^n \neq y^m$?

By the *abc* conjecture for each $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$|x^n - y^m| > c(\varepsilon) X^{1-\frac{1}{m}-\frac{1}{n}-\varepsilon}$$

where

$$X = \max(x^n, y^m).$$

Unconditionally Sprindzuk and Schmidt:

$$\exists\, \delta = \delta(m, n) > 0$$

such that

$$|x^m - y^n| \gg (\log X)^{\delta}.$$

Joint work with Frits Beukers

Conjecture 1.1

Let m, n *be coprime integers with* $n > m \geq 2$. *Then, for any* $c > 0$ *there exist infinitely many positive integers* x, y *such that*

$$0 < |x^n - y^m| < cX^{1-1/m-1/n},$$

where $X = \max(x^n, y^m)$.

Given m, n we define the number $\theta(m,n)$ as the infimum over all θ such that $0 < |x^n - y^m| < X^\theta$, $X = \max(x^n, y^m)$ has infinitely many solutions in positive integers x, y. We believe that $\theta(m,n) = 1 - 1/n - 1/m$. Only the trivial lower bound 0 is known, so we focus our attention on upper bounds. We define the number $\gamma(m,n)$ by

$$\theta(m,n) = 1 - (1/m) - (1/n) + \gamma(m,n)/mn$$

and find upper bounds for $\gamma(m,n)$. Our conjecture concerning $\theta(m,n)$ is now equivalent to

Conjecture 1.2

For any coprime integers m, n with $n > m \geq 2$ we have $\gamma(m,n) = 0$.

For any real number x let $[x]$ denote the greatest integer less than or equal to x.

Theorem 1.3

Let n and m be coprime integers with $n > m \geq 2$. We have

$$\gamma(m, n) \leq \min\{n - m[n/m], m - (n/([n/m] + 1))\}.$$

If n and m are both odd integers and n is slightly larger than an odd multiple of m we are able to improve on Theorem 1.3.

Theorem 1.4

Let n and m be coprime odd integers with $n > m \geq 2$. Write $n = (2q+1)m + w$ with $q \geq 0$ and $0 < |w| < m$. Then

$$\gamma(m,n) \leq \frac{n(|w|-1)-m}{n-1}.$$

One can check that Theorem 1.4 yields an improvement on Theorem 1.3 precisely when w is positive and $m > (2q+3)(w-1)$.

Here is a table of values for the upper bounds for $\gamma(m,n)$ given by Theorems 1.3 and 1.4. We appealed to Theorem 1.4 only for the entries $(5,7)$, $(7,9)$ and $(9,11)$.

m n	2	3	4	5	6	7	8	9	10
3	1/2								
4	*	1							
5	1/3	1/2	1						
6	*	*	*	1					
7	1/4	2/3	1/2	1/3	1				
8	*	1/3	*	1	*	1			
9	1/5	*	1	1/2	*	1/4	1		
10	*	1/2	*	*	*	2	*	1	
11	1/6	1/4	1/3	1	1/2	3/2	5/2	1/5	1
12	*	*	*	1	*	1	*	*	*
13	1/7	2/5	3/4	2/3	1	1/2	3/2	5/2	3
14	*	1/5	*	1/3	*	*	*	2	*
15	1/8	*	1/4	*	*	1	1/2	*	*
16	*	1/3	*	1	*	5/3	*	1	*
17	1/9	1/6	3/5	3/4	1/3	4/3	1	1/2	3/2

The proofs of Theorems 1.3 to 1.6 depend upon polynomial constructions and as a consequence for each pair (m, n) there are positive numbers $c_1 = c_1(m, n)$, $c_2 = c_2(m, n)$ and $\lambda = \lambda(m, n)$ such that for each positive integer N larger than c_2 there are at least N^λ pairs (x^n, y^m) with $1 \leq x^n \leq N$ for which

$$0 < |x^n - y^m| < cX^\theta.$$

We are able to improve on Theorems 1.3 and 1.4 for a sparse but infinite set of pairs (m, n).

Theorem 1.5

Let n and m be coprime positive integers with $n > m \geq 2$. Suppose that

$$\frac{6}{5} < \frac{n}{m} < \frac{3}{2}$$

and that there are positive integers u and v for which either $(m, n) = (6v^2 - u^2, 9v^2 - 2u^2)$ or $(m, n) = (2v^2 - 3u^2, 3v^2 - 6u^2)$ then

$$\gamma(m, n) \leq m - \frac{2n}{3}.$$

Theorem 1.5 yields an improvement of Theorem 1.3 whenever it is applicable.

We are also able to improve on Theorem 1.3 when the conditions of the next result apply.

Theorem 1.6

Let n and m be coprime positive integers with $n > m \geq 2$. Suppose that there is a rational number a such that $(x, y) = (a, n/m)$ is a point on the curve

$$E : x^3 + 3(y-3)x^2 + (y-3)(y-4)x + \frac{(y-3)(y-4)(y-5)}{15} = 0.$$

Then

$$\gamma(m,n) \leq n - \frac{5m}{2} \qquad \text{if} \quad \frac{5}{2} < \frac{n}{m} < \frac{21}{8}$$

and

$$\gamma(m,n) \leq m - \frac{2n}{5} \qquad \text{if} \quad \frac{15}{7} < \frac{n}{m} < \frac{5}{2}.$$

The curve E together with a rational point, such as $(0, 3)$, is an elliptic curve. It has Weierstrass form

$$Y^2 = X^3 - 2475X - 5850. \tag{1}$$

The set of real points (X, Y) on (1) consists of two connected components, one of which is unbounded. Since $(235, 3520)$, a non-torsion rational point, is in the latter component the rational solutions are dense in that component. Thus one may check that there are infinitely many coprime pairs of positive integers (m, n) for which $5/2 < n/m < 21/8$ and for which $15/7 < n/m < 5/2$. However there are only two pairs (m, n) with $m < 10,000$ for which n/m is in the above ranges. They are $(23, 59)$ and $(7991, 19980)$ and we have $\gamma(23, 59) \leq 3/2$ and $\gamma(7991, 19980) \leq 5/2$.

Theorem 1.7

$\gamma(11, 28) \leq 1/2$.

Our strategy for proving Theorems 1.3, 1.4, 1.5, 1.6 and 1.7 is the same as that employed by Birch, Chowla, Hall and Schinzel. That is, we look for polynomials f and g with rational coefficients and degrees mk and nk respectively for which $f^n - g^m$ is a non-zero polynomial of small degree and then we specialize to produce an m-th power of an integer and an n-th power of an integer which are close. Davenport has shown that there are limitations on how small the degree of $f^n - g^m$ can be. He proved that if f and g are non-constant polynomials in $\mathbb{C}[x]$ then either $f^n = g^m$ or

$$\deg(f^n - g^m) \geq kmn - km - kn + 1. \tag{2}$$

We shall call pairs (f^n, g^m) of polynomials with $f, g \in \mathbb{Q}[x]$ for which equality holds in (2) Davenport pairs.

Our approach to construct very close pairs of *m*-th and *n*-th powers is via polynomials. More particularly, it rests on the following Lemma.

Lemma 3.1

Let m, n be coprime integers with $n > m \geq 2$ and let k be a positive integer. Suppose there exist polynomials $f, g, h \in \mathbb{Q}[x]$ of degrees km, kn, D respectively with $0 < D < kmn$ such that

$$f^n - g^m = h.$$

Put $D = kmn - km - kn + \delta$. Then $\theta(m, n) \leq (1 - 1/m - 1/n) + \delta/kmn$ and $\gamma(m, n) \leq \delta/k$.

Proof.

Let N be the common denominator of the coefficients of the polynomials f, g. Then

$$(N^m f(x))^n - (N^n g(x))^m = N^{mn} h(x)$$

is an identity between polynomials with integer coefficients. We construct close m-th and n-th powers by substitution of x by an integer a. Since there exist positive constants a_0, b_0 and c_0 such that

$$(N^m f(a))^n / a^{kmn} \to a_0, \; (N^n g(a))^m / a^{kmn} \to b_0, \; N^{mn} h(a)/a^D \to c_0$$

as $a \to \infty$, our estimate for $\theta(m, n)$ follows. $\square$

To prove Theorem 1.3 we use the following polynomial constructions.

Lemma 3.2

Let m, n be coprime integers with $n > m \geq 2$ and let $s = [n/m]$. Let

$$B_1(x) = \sum_{j=0}^{s} \binom{n/m}{j} x^{n-jm}.$$

Then $(x^m + 1)^n - B_1(x)^m$ is a polynomial of degree $mn - m(s+1)$.

Lemma 3.3

Let m, n be coprime integers with $n > m \geq 2$ and let $s = [n/m] + 1$. Let

$$B_2(x) = \sum_{j=0}^{s} \binom{n/m}{j} x^{sn-jn}.$$

Then $(x^{ms} + x^{ms-n})^n - B_2(x)^m$ is a polynomial of degree $mns - n(s+1)$.

Proof of Lemmas 3.2 and 3.3.

Consider the following Taylor expansion in t,

$$(1+t)^{n/m} = \sum_{j=0}^{s} \binom{n/m}{j} t^j + O(t^{(s+1)}). \tag{3}$$

Replace t by $1/x^m$, raise both sides to the power m, and multiply on both sides by x^{mn}. We obtain

$$(x^m+1)^n = B_1(x)^m + O(x^{mn-m(s+1)}).$$

A more careful analysis of the constant in the O-term shows that the degree is precisely $mn - m(s+1)$ and Lemma 3.2 follows.

Proof of Lemmas 3.2 and 3.3 (continued).

For the proof of Lemma 3.3 we replace t by $1/x^n$ in (3), raise both sides to the power m and multiply on both sides by x^{mns}. We obtain

$$(x^{ms} + x^{ms-n})^n = B_2(x)^m + O(x^{mns-n(s+1)}).$$

A more careful analysis of the constant in the O-term shows that the degree is precisely $mns - n(s+1)$ and so Lemma 3.3 holds. ☐

The proof of Theorem 1.3 now goes as follows. Application of Lemma 3.1 with $k = 1$ and Lemma 3.2 gives us

$$\gamma(m, n) \leq mn - m([n/m] + 1) - mn + m + n = n - [n/m]m.$$

Application of Lemma 3.1 with $k = s$ and Lemma 3.3 gives us

$$\gamma(m, n) \leq mn - n(1 + 1/([n/m] + 1)) - mn + m + n = m - (n/([n/m] + 1)).$$

Put

$$\theta = \frac{x + \sqrt{x^2 - 4}}{2}$$

and notice that

$$\theta^{-1} = \frac{x - \sqrt{x^2 - 4}}{2}.$$

Define

$$T_n(x) = \theta^n + \theta^{-n}.$$

By putting $x = 2\cos\phi$ one sees that $\theta = e^{i\phi}$ and $T_n(2\cos\phi) = 2\cos n\phi$.

Let n and m be coprime integers with $n > m \geq 2$ and let q and w be the integers with $q \geq 0$ and $0 < |w| < m$ for which $n = (2q+1)m + w$. Notice that

$$(\theta^m + \theta^{-m})^{n/m} = \sum_{r=0}^{q} \binom{n/m}{2r} \theta^{n-2rm} + O(\theta^{-m+w})$$

and so

$$(T_m(x))^{n/m} = \sum_{r=0}^{q} \binom{n/m}{2r} T_{n-2rm}(x) + O(\theta^{-m+|w|}),$$

hence

$$(T_m(x))^n = \left(\sum_{r=0}^{q} \binom{n/m}{2r} T_{n-2rm}(x) \right)^m + O\left(x^{n(m-1)-m+|w|} \right). \tag{4}$$

If n is odd then all the non-zero coefficients of T_n are associated with odd powers of x while if n is even all the non-zero coefficients of T_n are associated with even powers of x. We now divide both sides of (4) by x^{nm} and put $t = x^{-2}$ to get

$$(1+A_1t+\cdots+A_Mt^M)^n = (1+B_1t+\cdots+B_Nt^N)^m+O(t^{(m+n-|w|)/2}), \tag{5}$$

where

$$M = \left[\frac{m}{2}\right] \quad \text{and} \quad N = \left[\frac{n}{2}\right]$$

and $A_1, \ldots, A_M$ and $B_1, \ldots, B_N$ are rational numbers which are the non-zero coefficients of $T_m(x)$ and $\sum_{r=0}^{q} \binom{n/m}{2r} T_{n-2rm}(x)$ respectively.

Proof of Theorem 1.4

Suppose that m and n are odd. Put $t = x^{-n}$ and multiply both sides of (5) by x^{mnN} to get

$$\begin{aligned}(x^{mN} + A_1 x^{mN-n} + \cdots + A_M x^{mN-Mn})^n \\ = (x^{nN} + B_1 x^{n(N-1)} + \cdots + B_N)^m + O(x^{mnN-n(m+n-|w|)/2}).\end{aligned}$$

Put

$$f(x) = x^{mN} + A_1 x^{mN-n} + \cdots + A_M x^{mN-Mn}$$

and

$$g(x) = x^{nN} + B_1 x^{n(N-1)} + \cdots + B_N.$$

Since $mN - Mn = (n - m)/2$ we see that f is a polynomial.

Further one may check that the degree D of $f(x)^n - g(x)^m$ is $mnN - n(m+n-|w|)/2$. Therefore we may apply Lemma 3.1 with $k = N$ to conclude that

$$\begin{aligned}\gamma(m,n) &\leq \frac{\left((m+n)N - \frac{n(m+n-|w|)}{2}\right)}{N} \\ &\leq m+n-\frac{n}{n-1}(m+n-|w|) = \frac{n(|w|-1)-m}{n-1}.\end{aligned}$$

Further improvements on Theorem 1.3

In this section we generalise the constructions employed in the proof of Lemmas 3.2 and 3.3. Instead of $(1+t)^{n/m}$ we consider

$$(1+t+at^2)^{n/m} = 1 + b_1 t + b_2 t^2 + \cdots + b_k t^k + b_{k+1} t^{k+1} + \cdots .$$

Now assume that we can find $a \in \mathbb{Q}$ such that $b_{k+1} = 0$. We get

$$(1+t+at^2)^{n/m} = 1 + b_1 t + b_2 t^2 + \cdots + b_k t^k + O(t^{k+2}). \quad (6)$$

We consider two possibilities for construction, depending on whether $2n - km \geq 0$ or $2n - km \leq 0$.

In the first case, when $2n - km \geq 0$, we replace t by $1/x^m$, raise everything to the power m and multiply by x^{2mn}. This yields

$$(x^{2m}+x^m+a)^n = (x^{2n}+b_1x^{2n-m}+\cdots+b_kx^{2n-km})^m+O(x^{2mn-(k+2)m}).$$

Using Lemma 3.1 we get the bound $\gamma(m,n) \leq n - mk/2$. This is an improvement over the bound given by Theorem 1.3 if and only if k is odd and

$$\frac{k}{2} < \frac{n}{m} < ((k+1)(k+2))/(2(k+3)).$$

In the second case, when $2n - km \leq 0$, we replace t by $1/x^n$, raise everything to the power m and multiply by x^{kmn}. We obtain

$$\begin{aligned}(x^{km} + x^{km-n} + ax^{km-2n})^n \\ = (x^{kn} + b_1x^{kn-n} + \cdots + b_k)^m + O(x^{kmn-(k+2)n}).\end{aligned}$$

Using Lemma 3.1 we get the bound $\gamma(m, n) \leq m - 2n/k$. This is an improvement over Theorem 1.3 if and only if k is odd and $(k(k+1))/(2(k+2)) < n/m < k/2$. Everything we said, of course, relies on our success in finding a rational number a for which $b_k = 0$.

The case $k = 3$ and the proof of Theorem 1.5

A brief calculation shows that $b_4 = 0$ implies

$$12a^2m^2 - 24am^2 + 12amn + 6m^2 - 5mn + n^2 = 0. \tag{7}$$

This is quadratic in a, so a is rational if and only if the discriminant

$$48m^2(3m - 2n)(2m - n)$$

is a square. Since $\gcd(m, n) = 1$ there are four possibilities:

1. $2n - 3m = u^2$ and $n - 2m = 3v^2$ for some coprime positive integers u, v. Hence $m = -6v^2 + u^2$ and $n = -9v^2 + 2u^2$.
2. $2n - 3m = 3u^2$ and $n - 2m = v^2$ for some coprime positive integers u, v. Hence $m = -2v^2 + 3u^2$ and $n = -3v^2 + 6u^2$.
3. $3m - 2n = u^2$ and $2m - n = 3v^2$ for some coprime positive integers u, v. Hence $m = 6v^2 - u^2$ and $n = 9v^2 - 2u^2$.
4. $3m - 2n = 3u^2$ and $2m - n = v^2$ for some coprime positive integers u, v. Hence $m = 2v^2 - 3u^2$ and $n = 3v^2 - 6u^2$.

In the first two cases we have $2n - 3m \geq 0$ and so we obtain an improvement of Theorem 1.3 when $3/2 < n/m < 5/3$. However, one can check that this does not occur. In the next two cases $2n - 3m < 0$ and we obtain an improvement of Theorem 1.3 when $6/5 < n/m < 3/2$. In case 3 we see that this is equivalent to the condition $2u < 3v$ while in case 4 it is equivalent to the condition $2u < v$. Further in both cases, by our earlier remarks, $\gamma(m, n) \leq m - 2n/3$ and so Theorem 1.5 follows.

The case $k = 5$ and the proof of Theorem 1.6

The equation $b_6 = 0$ is equivalent to

$$15a^3 + 45\left(\frac{n}{m} - 3\right)a^3 + 15\left(\frac{n}{m} - 3\right)\left(\frac{n}{m} - 4\right)a + \left(\frac{n}{m} - 3\right)\left(\frac{n}{m} - 4\right)\left(\frac{n}{m} - 5\right) = 0.$$

We have an improvement of Theorem 1.3 if $5/2 < n/m < 21/8$ in which case $\gamma(m, n) \leq n - 5m/2$ or if $15/7 < n/m < 5/2$ in which case $\gamma(m, n) \leq m - 2n/5$. Theorem 1.5 now follows.

The proof of Theorem 1.7

One may check that

$$\Big((x^{22}+11x^{11}+22)^{28},$$
$$(x^{56}+28x^{45}+294x^{34}+1428x^{23}+3213x^{12}+2856x)^{11}\Big) \tag{8}$$

is a Davenport pair. It now follows from Lemma 3.1 with $\delta = 1$ and $k = 2$ that $\gamma(11, 28) \leq 1/2$.

Infinite families of Davenport pairs

In this section we list the Davenport pairs (f^n, g^m) with f and g in $\mathbb{Q}[x]$ which we have found. Suppose that the degree of f is km and the degree of g is kn with k a positive integer. We consider the pairs $(f(x)^n, g(x)^m)$ and $(c^{km} f(ax+b)^n, (c^{kn} g(ax+b))^m)$ with $a, b, c \in \overline{\mathbb{Q}}$, $ac \neq 0$ as equivalent. Accordingly we may choose a representative (f^n, g^m) from each equivalence class with f a monic polynomial of degree km having 0 as the coefficient of degree $km-1$ and with the next non-zero coefficient an integer of smallest absolute value which, when possible, is taken to be positive. We shall refer to this as the normalized form for a representative.

We have found seven infinite families of Davenport pairs as well as a number of sporadic examples. Two of the families arise from the Taylor series expansion of $(1+t)^{n/m}$, three from the Taylor series expansion of $(1+t+at^2)^{n/m}$ and the two other infinite families are connected with the Chebyshev polynomials.

Notice that if $n - sm = 1$ for some positive integer s, or equivalently that $m \mid n - 1$ then by Lemma 3.2

$$((x^m + 1)^n, B_1(x)^m) \tag{9}$$

is a Davenport pair. Recall that

$$B_1(x) = \sum_{j=0}^{s} \binom{n/m}{j} x^{n-jm},$$

where $s = [n/m]$.

Proposition 7.1

Let n and m be coprime positive integers with $n > m \geq 2$. Suppose that there is a positive divisor d of n such that m divides $d+1$. Put $s = (d+1)/m$ and

$$B_3(x) = \sum_{j=0}^{ns/d} \binom{n/m}{j} x^{ns-jd}.$$

Then $(x^{d+1}+x)^n - B_3^m$ has degree $mns - ms - ns + 1$.

It follows from Proposition 7.1 that if $d > 0$, $d \mid n$ and $m \mid d+1$ then

$$\left((x^{d+1}+x)^n, B_3^m(x)\right) \tag{10}$$

is a Davenport pair.

$$\left(\left(x^{2d} + x^{d} + \frac{2d-1}{12d}\right)^{2d+1}, \right. \\ \left. \left(x^{2d+1} + \frac{2d+1}{2d}x^{d+1} + \frac{2d^2+3d+1}{12d^2}x\right)^{2d} \right) \tag{11}$$

is a Davenport pair for d a positive integer.

We have, for e in $\{1, -1\}$,

$$\left(f_e^{3v^2-6}, g_e^{2v^2-3}\right) \tag{12}$$

is a Davenport pair with

$$f_e(x) = x^{2v^2-3} + x^{v^2-1} + \frac{v(v+e)}{2(2v^2-3)}x$$

and

$$\begin{aligned} g_e(x) = x^{3v^2-6} &+ \frac{3v^2-6}{2v^2-3}x^{2v^2-4} \\ &+ \frac{3(v^2-2)(2v^2+ev-3)}{2(2v^2-3)^2}x^{v^2-2} \\ &+ \frac{(v^2-2)v(v^2-3)(2v+3e)}{2(2v^2-3)^3} \end{aligned}$$

for $v = 2, 3, \ldots$.

We have, for e in $\{1, -1\}$,

$$\left(f_e^{9v^2-2}, g_e^{6v^2-1}\right) \tag{13}$$

is a Davenport pair with

$$f_e(x) = x^{18v^2-3} + x^{9v^2-1} + \frac{v(3v+e)}{2(6v^2-1)}x$$

and

$$\begin{aligned} g_e(x) = x^{27v^2-6} &+ \frac{9v^2-2}{6v^2-1}x^{18v^2-4} \\ &+ \frac{(9v^2-2)(6v^2+ev-1)}{2(6v^2-1)^2}x^{9v^2-2} \\ &+ \frac{(9v^2-2)v(3v^2-1)(2v+e)}{2(6v^2-1)^3} \end{aligned}$$

for $v = 1, 2, \ldots$.

We remark that the representatives given for the equivalence classes of Davenport pairs for the families (9), (10), (11), (12) and (13) are in normalized form with the exception of the case $d = 1$ in (10) and the case $d = 1$ in (11). Further we may take f_e and g_e in non-normalized form to be

$$f_e(x) = x^{2v^2-3} + (2v^2-3)x^{v^2-1} + \frac{v(v+e)(2v^2-3)}{2}x$$

and

$$\begin{aligned} g_e(x) = x^{3v^2-6} + (3v^2-6)x^{2v^2-4} \\ + \frac{3(v^2-2)(2v^2+ev-3)}{2}x^{v^2-2} \\ + \frac{(v^2-2)v(v^2-3)(2v+3e)}{2} \end{aligned}$$

in (12)

and to be

$$f_e(x) = x^{18v^2-3} + (6v^2-1)x^{9v^2-1} + \frac{v(3v+e)(6v^2-1)}{2}x$$

and

$$\begin{aligned} g_e(x) = x^{27v^2-6} + (9v^2-2)x^{18v^2-4} \\ + \frac{(9v^2-2)(6v^2+ev-1)}{2}x^{9v^2-2} \\ + \frac{(9v^2-2)v(3v^2-1)(2v+e)}{2} \end{aligned}$$

in (13).

The Chebyshev families

Let m, q and e be integers with $m \geq 2$, $q \geq 0$ and e from $\{1, -1\}$. Put $n = (2q+1)m + e$. Then, by (4),

$$\left((T_m(x))^n, \left(\sum_{r=0}^{q} \binom{n/m}{2r} T_{n-2rm}(x) \right)^m \right) \tag{14}$$

is a Davenport pair.

Further by taking m odd and $n = m + 2$ in §4.1 we see that

$$\left(f_m^{m+2}, g_{m+2}^{m}\right) \tag{15}$$

is a Davenport pair where

$$f_m(x) = x^{-m/2} T_m(x^{(m+2)/2})$$

and

$$g_{m+2}(x) = x^{-m+2/2} T_{m+2}(x^{(m+2)/2}).$$

The first few polynomials $T_m(x)$ are

$$
\begin{aligned}
T_2(x) &= x^2 - 2\\
T_3(x) &= x^2 - 3x\\
T_4(x) &= x^4 - 4x^2 + 2\\
T_5(x) &= x^5 - 5x^3 + 5x\\
T_6(x) &= x^6 - 6x^4 + 9x^2 - 2\\
T_7(x) &= x^7 - 7x^5 + 14x^3 - 7x
\end{aligned}
$$

and in normalized form they are $t_m(x)$ where

$$\begin{aligned}
t_2(x) &= x^2 + 1 \\
t_3(x) &= x^3 + x \\
t_4(x) &= x^4 + x^2 + \frac{1}{8} \\
t_5(x) &= x^5 + x^3 + \frac{x}{5} \\
t_6(x) &= x^6 + x^4 + \frac{x^2}{4} + \frac{1}{108} \\
t_7(x) &= x^7 + x^5 + \frac{2}{7}x^3 + \frac{1}{49}x.
\end{aligned}$$

Davenport pairs with $(m, n) = (2, 3)$

The case when $(m, n) = (2, 3)$ has been studied intensively. It is readily checked that when k is $1,\ 2$ or 3 there is only one equivalence class of solutions with coefficients in $\mathbb{Q}$. When $k = 1,\ f = x^2 + 1,\ y = x^3 + (3/2)x$ is a representative solution, when $k = 2$ we may take

$$f = x^4 + x^2 + \frac{1}{4}, \quad g = x^6 + \frac{3}{2}x^4 + \frac{3}{4}x^2 + \frac{1}{8}$$

and when $k = 3$ we may take

$$f = x^6 + x^4 + \frac{5}{8}x^2 + \frac{3}{32}, \quad g = x^9 + \frac{3}{2}x^7 + \frac{21}{16}x^5 + \frac{35}{64}x^3 + \frac{63}{512}x.$$

Hall found an example with $k = 4$. Normalized in the usual manner the example is

$$\begin{aligned} f &= x^8 + 21x^6 + 22x^5 + \frac{1183}{8}x^4 + 423x^3 + \frac{6721}{16}x^2 \\ &\quad + \frac{13679}{8}x + \frac{268777}{256} \\ g &= x^{12} + \frac{63}{2}x^{10} + 33x^9 + \frac{6195}{16}x^8 + 981x^7 + \frac{42339}{16}x^6 \\ &\quad + \frac{78783}{8}x^5 + \frac{3758439}{256}x^4 + \frac{632675}{16}x^3 + \frac{32269011}{512}x^2 \\ &\quad + \frac{13826697}{256}x + \frac{280013653}{4096}, \end{aligned}$$

and it follows, for example, from work of Zannier that there is only one equivalence class with coefficients $\mathbb{Q}$ with $(n, m, k) = (3, 2, 4)$. Zannier relates equivalence classes of solutions over $\mathbb{C}$ to certain regular trees. His approach is connected with the dessins d'enfants of Grothendieck. Similarly one can show that there are at most two equivalence classes of solutions over $\mathbb{Q}$ for $k = 5$.

Birch, Chowla, Hall and Schinzel found one example

$$
\begin{aligned}
f &= x^{10} + x^7 + \frac{5}{12}x^4 + \frac{1}{18}x \\
g &= x^{15} + \frac{3}{2}x^{12} + x^9 + \frac{1}{3}x^6 + \frac{5}{96}x^3 + \frac{1}{576}.
\end{aligned}
$$

Further Elkies found the example

$$\begin{aligned} f &= x^{10} - 2x^9 + 33x^8 - 12x^7 + 378x^6 + 336x^5 + 2862x^4 \\ &\quad + 2652x^3 + 14397x^2 + 9922x + 18553 \\ g &= x^{15} - 3x^{14} + 51x^{13} - 67x^{12} + 969x^{11} + 33x^{10} + 10963x^9 \\ &\quad + 9729x^8 + 96507x^7 + 108631x^6 + 580785x^5 + 700503x^4 \\ &\quad + 2102099x^3 + 1877667x^2 + 3904161x + 1164691, \end{aligned}$$

which we have not normalized in the usual manner.

Therefore the complete list of Davenport pairs with $(n, m) = (3, 2)$ and k at most 5 is known. It is not known if there exist any with k larger than 5. This question was posed already by Birch, Chowla, Hall and Schinzel in 1965.

Some sporadic Davenport pairs

We remark that if (n, m, k) is specified with $n > m$ and (f^n, g^m) is a Davenport pair then the coefficients of g are determined once the coefficients of f are known.

Since the pairs are determined by f when n, m and k are given we need only list f. We have found the pairs by means of the Groebner package in MAPLE.

(m, n, k)	f
$(2, 5, 2)$	$x^4 + 6x^2 + 64x - 55$
$(3, 5, 1)$	$x^3 + x + \frac{1}{3}$
$(3, 7, 1)$	$x^3 + 2x + \frac{2}{3}$
$(3, 8, 1)$	$x^3 + 3x + 3$
$(3, 10, 1)$	$x^3 + 6x + 6$
$(4, 5, 1)$	$x^4 - 2x^2 + 2x + \frac{3}{2}$
$(5, 6, 1)$	$x^5 + x^3 + \frac{3}{5}x$
$(5, 9, 1)$	$x^5 + 2x^3 + \frac{4}{5}x + \frac{4}{25}$
$(5, 11, 1)$	$x^5 + 3x^3 + \frac{9}{5}x + \frac{9}{25}$
$(5, 14, 1)$	$x^5 + 6x^3 + \frac{36}{5}x + \frac{108}{25}$
$(5, 16, 1)$	$x^5 + x^3 + \frac{x}{5} + \frac{1}{25}$
$(11, 28, 2)$	$x^{22} + x^{11} + \frac{2}{11}$