Moment Matrices, Trace Matrices and the Radical of Ideals

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The problem

Given: $f_1, \ldots, f_s \in \mathbb{C}[\mathbf{x}]$ polynomials in $\mathbf{x} = (x_1, \ldots, x_m)$ generating an ideal I.
Assume that I has finitely many roots in $\mathbb{C}^m$.
Suppose I either has roots with multiplicities or form clusters with radius $\varepsilon > 0$.

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We propose here to use **Sylvester matrices** to compute matrices of traces.

Related previous work

- Global methods for approximate square-free factorization (univariate case): Sasaki and Noda (1989), Hribernig and Stetter (1997), Kaltofen and May (2003), Zeng (2003), Corless, Watt and Zhi (2004).

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- Local methods to handle near root multiplicities
  - Using eigenvalue computations: Manocha and Demmel (1995), Corless, Gianni and Trager (1997).
  - Using deflation: Ojica, Watanabe and Mitsui (1983), Ojica (1987), Lecerf (2002), Leykin, Verschelde and Zhao (2005).
  - Using dual bases: Stetter (1996) and (2004), Dayton and Zeng (2005).

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- Computing the matrix of traces from the generating polynomials of I: Becker, Cardinal, Roy, Szafraniec (1994), Cardinal and Mourrain (1996), Cattani, Dickenstein and Sturmfels (1996) and (1998), Mourrain and Pan (2000), Díaz-Toca and González-Vega (2001), Briand and González-Vega (2001), D'Andrea and Jeronimo (2005).

Multiplication matrices

Definition

Let $I = \langle f_1, \ldots, f_s \rangle$ be and ideal for which $A = \mathbb{C}[\mathbf{x}]/I$ is finite dimensional. Let $B = [b_1, \ldots, b_n]$ be a basis of A. The multiplication matrix M_h is the transpose of the matrix of the map

$$m_h \; : \; A \to A, \quad [g] \mapsto [hg]$$

written in the basis B.

Univariate Example

Example

Let

$$f = x^n + a_1 x^{n-1} \cdots a_{n-1}x + a_n \in \mathbb{C}[x]$$

be a monic polynomial, and $B := [1, x, x^2, \ldots, x^{n-1}]$ be a basis for $\mathbb{C}[x]/\langle f \rangle$.

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$$M_x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

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If $f = (x - \xi_1) \cdots (x - \xi_n)$ with ξ_i all distinct, then

$$M_x = V \,\mathrm{diag}(\xi_1, \ldots, \xi_n)\, V^{-1},$$

where V is the Vandermonde matrix of $\xi_1, \ldots, \xi_n$.

Expressions in the roots

Let $\mathbf{z}_1, \ldots, \mathbf{z}_n \in \mathbb{C}^m$ be the roots of I and $B = [b_1, \ldots, b_n]$ be a basis of $A = \mathbb{C}[\mathbf{x}]/I$. Define the Vandermonde matrix

$$V := [b_j(\mathbf{z}_i)]_{i,j=1}^{n} \in \mathbb{C}^{n \times n}.$$

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Fact

If V is invertible then

$$M_h = V \ \operatorname{diag}(h(\mathbf{z}_1), \ldots, h(\mathbf{z}_n)) \ V^{-1},$$

i.e. he multiplication matrices M_h are simultaneously diagonalizable with $h(\mathbf{z}_1), \ldots, h(\mathbf{z}_n)$ eigenvalues.

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Goal: Compute multiplication matrices for the radical $\sqrt{I}$.

Matrix of traces

Definition

Let $B = [b_1, \ldots, b_n]$ be a basis of $A = \mathbb{C}[\mathbf{x}]/I$. The matrix of traces is the $n \times n$ symmetric matrix:

$$R = [Tr(b_i b_j)]_{i,j=1}^{n}$$

where $Tr(b_i b_j)$ is the trace of the multiplication matrix $M_{b_i b_j}$.

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Fact

$$R = V \cdot V^T,$$

where $V := [b_j(\mathbf{z}_i)]_{i,j=1}^n$ is the Vandermonde matrix for the roots $\mathbf{z}_1, \ldots, \mathbf{z}_n \in \mathbb{C}^m$ of I. Moreover

$$\mathrm{rank}(R) = \#\{\text{ distinct roots of } I\} = \dim \mathbb{C}[\mathbf{x}]/\sqrt{I}.$$

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Idea: We will use a maximal non-singular submatrix of R to eliminate root multiplicities.

Dickson's Lemma

Theorem (Dickson (1923))

Let $B = [b_1, \ldots, b_n]$ *be a basis of* $A = \mathbb{C}[\mathbf{x}]/I$. *An element*

$$r = \sum_{k=1}^{n} c_k b_k$$

is in $\mathrm{Rad}(A) = \sqrt{I}/I$ *if and only if* $[c_1, \ldots, c_n]$ *is in the nullspace of the matrix of traces* R.

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Corollary

Let $R = [Tr(b_i b_j)]_{i,j=1}^{n}$ and define $R_{x_k} := [Tr(x_k b_i b_j)]_{i,j=1}^{n}$ for $k = 1, \ldots, m$.
If $\tilde{R}$ is a maximal non-singular submatrix of R, and $\tilde{R}_{x_k}$ is the submatrix of R_{x_k} with the same row and column indices as in $\tilde{R}$, then the solution $\tilde{M}_{x_k}$ of the linear matrix equation

$$\tilde{R}\tilde{M}_{x_k} = \tilde{R}_{x_k}$$

is a multiplication matrix of x_k for the radical of $\sqrt{I}$.

Clusters of roots

We consider systems for which the common roots form clusters of roots.

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Definition

Let $\mathbf{z}_i \in \mathbb{C}^m$ for $i = 1, \ldots, k$, and consider k clusters $C_1, \ldots, C_k$ of size $|C_i| = n_i$ such that $\sum_{i=1}^k n_i = n$, each of radius proportional to the parameter ε around $\mathbf{z}_1, \ldots, \mathbf{z}_k$:

$$C_i = \{\mathbf{z}_i + \delta_{i,1}\varepsilon, \ldots, \mathbf{z}_i + \delta_{i,n_i}\varepsilon\},$$

where all the coordinates of $\delta_{i,j}$ are less than 1 for all i, j.

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In this setting we will use trace matrices to define an approximate radical.

GECP and SVD for the matrix of traces

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Proposition

The U_k be the matrix obtained after k steps of the Gaussian Elimination with Complete Pivoting (GECP) on R for a system with k clusters is of the form

$$\begin{bmatrix} [U_k]_{1,1} & & \cdots & \cdots & \cdots & [U_k]_{1,n} \\ 0 & \ddots & \cdots & \cdots & \cdots & \vdots \\ & & [U_k]_{k,k} & \cdots & \cdots & [U_k]_{k,n} \\ \vdots & & 0 & c_{k+1,k+1}\varepsilon^2 & \cdots & c_{k+1,n}\varepsilon^2 \\ & & \vdots & \vdots & \ddots & \vdots \\ 0 & & 0 & c_{n,k+1}\varepsilon^2 & \cdots & c_{n,n}\varepsilon^2 \end{bmatrix} + h.o.t.(\varepsilon).$$

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Proposition

Let $\sigma_1 \geq \cdots \geq \sigma_n$ be the singular values of R. Then

$$\sigma_{k+1} = C\,\varepsilon^2 + h.o.t.(\varepsilon).$$

Multiplication matrices for the approximate radical

Definition

Let $\tilde{R}$ be a maximal numerically non-singular submatrix of R, and $\tilde{R}_{x_i}$ is the submatrix of R_{x_i} with the same row and column indices as in $\tilde{R}$. Then the solution $\tilde{M}_{x_i}$ of the linear matrix equation

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Theorem

Modulo ε^2 the multiplication matrices $\tilde{M}_{x_1}, \ldots, \tilde{M}_{x_m}$ form a pairwise commuting system of matrices for the roots $\xi_1, \ldots, \xi_k$ satisfying

$$\xi_s = \mathbf{z}_s + \frac{\sum_{r=1}^{n_s} \delta_{s,r}}{n_s}\varepsilon \pmod{\varepsilon^2}.$$

Example

Consider the polynomial system:

$$f_1 = x_1^2 + 3.99980x_1x_2 - 5.89970x_1 + 3.81765x_2^2 - 11.25296x_2 + 8.33521$$

$$f_2 = x_1^3 + 12.68721x_1^2x_2 - 2.36353x_1^2 + 81.54846x_1x_2^2 - 177.31082x_1x_2 + 73.43867x_1 - x_2^3 + 6x_2^2 + x_2 + 5$$

$$f_3 = x_1^3 + 8.04041x_1^2x_2 - 2.16167x_1^2 + 48.83937x_1x_2^2 - 106.72022x_1x_2 + 44.00210x_1 - x_2^3 + 4x_2^2 + x_2 + 3$$

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Roots: $[0.8999, 1], [1, 1], [1, 0.8999]$ and $[-1, 2], [-1.0999, 2]$.
$\varepsilon = 0.1$.

Example

Basis: $[1, x_1, x_2, x_1 x_2, x_1^2]$.

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The matrix of traces:

$$R = \begin{bmatrix} 5 & 0.79999 & 6.89990 & -1.40000 & 5.01960 \\ 0.79999 & 5.01960 & -1.40000 & 7.12928 & 0.39812 \\ 6.89990 & -1.40000 & 10.80982 & -5.68988 & 7.12928 \\ -1.40000 & 7.12928 & -5.68988 & 11.45876 & -2.03262 \\ 5.01960 & 0.39812 & 7.12928 & -2.03262 & 5.11937 \end{bmatrix}.$$

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After 2 steps of GECP:

$$U_2 = \begin{bmatrix} 11.45876 & -5.68988 & 7.12928 & -1.40000 & -2.03262 \\ 0 & 7.98449 & 2.14006 & 6.20472 & 6.11998 \\ 0 & 0 & 0.01039 & 0.00799 & 0.02243 \\ 0 & 0 & 0.00799 & 0.00728 & 0.01544 \\ 0 & 0 & 0.02243 & 0.01544 & 0.06796 \end{bmatrix}.$$

Example

From the matrix of traces R we compute the matrix $\tilde{R}$, with columns indexed by 1 and x_1 and rows indexed by 1 and $x2$:

$$\tilde{R} := \begin{bmatrix} 5 & 0.79999 \\ 6.89990 & -1.40000 \end{bmatrix}.$$

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We now solve the system:

$$\tilde{R}\tilde{M}_{x_i} = \tilde{R}_{x_i}, \quad \text{with}$$
$$\tilde{R}_{x_1} = \begin{bmatrix} 0.79999 & 5.01960002 \\ -1.40000 & 7.12928003 \end{bmatrix},$$
$$\tilde{R}_{x_2} = \begin{bmatrix} 6.8999 & -1.4000 \\ 10.80982 & -5.68988 \end{bmatrix}$$

Example

We obtain the *approximate multiplication matrices*, in the basis $\{1, x_1\}$:

$$\tilde{M}_{x_1} = \begin{bmatrix} 0 & 1.01685 \\ 1 & -0.08080 \end{bmatrix}, \quad \text{with eigenvalues } 0.96880 \text{ and } -1.04960,$$

$$\tilde{M}_{x_2} = \begin{bmatrix} 1.46229 & -0.52012 \\ -0.51442 & 1.50078 \end{bmatrix}, \quad \text{with eigenvalues } 0.96391 \text{ and } 1.99915.$$

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The roots of the approximate radical are then $[0.96880, 0.96391]$ and

$[-1.0460, 1.99915]$.

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The commutator of the multiplication matrices is

$$\begin{bmatrix} -0.00296 & -0.00289 \\ 0.00307 & 0.00296 \end{bmatrix}.$$

Computation of Matrices of Traces

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Brute Force

Compute the multiplication matrices $M_{x_1}, \ldots, M_{x_m}$ using a Sylvester matrix, and use that $M_h = h(M_{x_1}, \ldots, M_{x_m})$ to compute the traces $Tr(M_{b_i b_j})$ for all $b_i, b_j \in B$.

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Newton Sums

Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = \prod_{i=1}^n (x - \xi_i)$. We have $R = [s_{i+j}]_{i,j=0}^{n-1}$ where $s_k := \sum_{t=1}^n \xi_t^k$.

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$$s_1 + a_1 = 0$$
$$s_2 + a_1 s_1 + 2a_2 = 0$$
$$\vdots$$
$$s_{2n-2} + a_1 s_{2n-3} + \cdots + a_n s_{n-3} = 0.$$

Note that this has generalizations to the multivariate case, but complicated.

Ingredients of our method

Let $\mathbf{f} = \{f_1, \ldots, f_s\} \subset \mathbb{C}[\mathbf{x}]$ generating an ideal I.

- A degree Δ and the Sylvester matrix $\mathrm{Syl}_\Delta(\mathbf{f})$, corresponding the map $(g_1, \ldots, g_s) \mapsto \sum_{i=1}^{s} f_i g_i$.
  The degree Δ is large enough so that a basis $B = [b_1, \ldots, b_n]$ for $A = \mathbb{C}[\mathbf{x}]/I$ can be computed using $\mathrm{Syl}_\Delta(\mathbf{f})$.

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  The degree Δ is large enough so that a basis $B = [b_1, \ldots, b_n]$ for $A = \mathbb{C}[\mathbf{x}]/I$ can be computed using $\mathrm{Syl}_\Delta(\mathbf{f})$.
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- The matrix $\mathrm{Syl}_B(J)$, corresponding to the map $\sum_{i=1}^{N} c_i b_i \mapsto J \cdot \sum_{i=1}^{N} c_i b_i \ \in \mathbb{C}[x]_\Delta$.

Main Theorem

Theorem

Let $B = [b_1, \ldots, b_n]$ *be a basis of* A *with* $\deg(b_i) \leq \Delta$. *With the generalized Jacobian* J *and* $\mathrm{Syl}_B(J)$ *defined before, we have*

$$[Tr(b_i b_j)]_{i,j=1}^{n} = \mathrm{Syl}_B(J) \cdot X,$$

where X *is the unique extension of the moment matrix* $\mathfrak{M}_B(\mathbf{y})$ *such that* $\mathrm{Syl}_\Delta(\mathbf{f}) \cdot X = 0$.

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The resulting moment matrix $\mathfrak{M}_B(\mathbf{y})$ an the matrix X of the Theorem are:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_1 \\ 1 & -a_1 & {a_1}^2 - a_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_1 \\ 1 & -a_1 & {a_1}^2 - a_2 \\ -a_1 & {a_1}^2 - a_2 & -{a_1}^3 + 2\,a_2a_1 - a_3 \\ {a_1}^2 - a_2 & -{a_1}^3 + 2\,a_2a_1 - a_3 & {a_1}^4 - 3\,a_2{a_1}^2 + 2\,a_3a_1 + {a_2}^2 \end{bmatrix}.$$

Univariate example cont.

The generalized Jacobian in this case is $J := f' = 3x^2 + 2a_1x + a_2$, and its Sylvester matrix is

$$\mathrm{Syl}_B(f') = \begin{bmatrix} a_2 & 2\,a_1 & 3 & 0 & 0 \\ 0 & a_2 & 2\,a_1 & 3 & 0 \\ 0 & 0 & a_2 & 2\,a_1 & 3 \end{bmatrix}.$$

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Finally, we get that $\mathrm{Syl}_B(f') \cdot X$ is the matrix of traces R:

$$\begin{bmatrix} 3 & -a_1 & -2\,a_2 + {a_1}^2 \\ -a_1 & -2\,a_2 + {a_1}^2 & -3\,a_3 + 3\,a_2a_1 - {a_1}^3 \\ -2\,a_2 + {a_1}^2 & -3\,a_3 + 3\,a_2a_1 - {a_1}^3 & -4\,a_2{a_1}^2 + 2\,{a_2}^2 + {a_1}^4 + 4\,a_3a_1 \end{bmatrix}.$$

THANK YOU!